ASSIGNMENT II

Problem 1. In this problem, we will work on a d-regular graph $G = (V, E)$ with $|V| = n$. Let N be its normalized Laplacian and $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$ be its eigenvalues. In the class, we proved the Cheeger's inequality that relates λ_2 with $\phi(G)$. In this problem, we will develop a similar inequality for λ_n together. We use μ to denote $2 - \lambda_n$.

(1) Prove that

$$\mu = \min_{\mathbf{x} \in \mathbb{R}^n \setminus \{\mathbf{0}\}} \frac{\sum_{\{u,v\} \in E} (\mathbf{x}(u) + \mathbf{x}(v))^2}{d \sum_{u \in V} \mathbf{x}(u)^2}.$$

(2) Let us recall the intuition behind Cheeger's inequality for λ_2. We have shown that $\lambda_2 = 0$ iff the graph G is not connected, and small λ_2 implies that G is *close* to being disconnected, i.e., it contains a vertex set with small expansion. Similarly, we know that $\mu = 0$ (equivalently $\lambda_n = 2$) iff the graph G is bipartite and therefore small μ means that G is close to being bipartite.

To formalize what we called *close to being bipartite*, let us consider a set $S \subseteq V$ and a partition $A \sqcup B = S$. Define its *bipartiteness* as

$$\psi(S, A, B) \triangleq \frac{\sum_{u \in S} |E(u)| - 2|E(A, B)|}{d|S|},$$

where for a vertex u, the set $E(u)$ is the set of edges in G incident to u, and $E(A, B)$ is the set of edges in G between A and B. We can also use a vector $\mathbf{y} \in \{-1, 0, 1\}^V$ to express the tuple (S, A, B) in which for every $u \in V$

$$\mathbf{y}(u) = \begin{cases} -1, & \text{if } u \in A; \\ 1, & \text{if } u \in B; \\ 0, & \text{if } u \notin S. \end{cases}$$

We can then define

$$\psi(\mathbf{y}) \triangleq \frac{\sum_{\{u,v\} \in E} |\mathbf{y}(u) + \mathbf{y}(v)|}{d \sum_{u \in V} |\mathbf{y}(u)|}.$$

Now assuming that $\mathbf{y}$ encodes a tuple (S, A, B), please prove that

$$\psi(\mathbf{y}) = \psi(S, A, B).$$

(3) We can now naturally define the *bipartiteness ratio* of G as

$$\psi(G) \triangleq \min_{\mathbf{y} \in \{-1,1,0\}^V \setminus \{\mathbf{0}\}} \psi(\mathbf{y}).$$

Prove that

$$\frac{\mu}{2} \le \psi(G).$$

(4) Consider the following bipartite version of Fiedler's algorithm:

> FIEDLER'S ALGORITHM
> Input: A graph $G = (V, E)$ and a vector $\mathbf{x} \in \mathbb{R}^n$.
> 1. For every $i \in V$, define $A_i = \{ j \in V \mid \mathbf{x}(j) \le -|\mathbf{x}(i)| \}$, $B_i = \{ j \in V \mid \mathbf{x}(j) \ge |\mathbf{x}(i)| \}$ and $S_i = A_i \cup B_i$.
> 2. Return $\min_{1 \le i \le n} \psi(S_i, A_i, B_i)$.

Let (S, A, B) be the tuple returned by Fiedler's algorithm on an input $(G, \mathbf{x})$. Prove that

$$\psi(S, A, B) \le \sqrt{2 \cdot \frac{\sum_{\{i,j\} \in E} (\mathbf{x}(i) + \mathbf{x}(j))^2}{d \sum_{i \in V} \mathbf{x}(i)^2}},$$

and conclude with

$$\psi(G) \leq \sqrt{2\mu}.$$

Problem 2. In Section 3 of Lecture 7 (Apr 08), I introduced the linear programming dual of the multi-commodity flow problem. Please prove that the dual is equivalent to Leighton-Rao relaxation of sparsest cut.