ASSIGNMENT III

Problem 1. Prove the following statement. If P is the transition matrix of an irreducible and aperiodic chain on the state space Ω, then there exists some $t \geq 1$ such that $P^t(x, y) > 0$ holds for every $x, y \in \Omega$.

Problem 2. Prove the following statement. Let P be the transition matrix of a reversible Markov chain on $[n]$ with stationary distribution π. Let $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$ be its eigenvalues. Then

(1) $\lambda_n = 1$;
(2) $\lambda_1 \geq -1$ and $\lambda_1 = -1$ if and only if one of components of $\mathcal{G}_P$ is bipartite;
(3) $\lambda_{n-1} = 1$ if and only if P is reducible.

Problem 3. Let p be the transition matrix of an irreducible and aperiodic chain with stationary distribution π . Let μ be an arbitrary distribution. Prove that the distance

$$d_{\mathrm{TV}}(\mu^T P^t, \pi)$$

is non-increasing in t.

Problem 4. Recall in Lecture XII we use P_i to denote a chain on Ω_i for every $i = 1, \ldots, n$ respectively. We also define the *product chain* P on $\Omega_1 \times \Omega_2 \times \cdots \times \Omega_n$ as follows: when one is standing at a point $\mathbf{x} \in \Omega$,

- choose an index $i \in [n]$ uniformly at random; and then
- perform a move in P_i.

Equivalently, we have for every $\mathbf{x}, \mathbf{y} \in \{0, 1\}^n$,

$$P(\mathbf{x}, \mathbf{y}) = \sum_{i=1}^{n} \frac{1}{n} \cdot P_i(x_i, y_i) \cdot \prod_{j \neq i} \mathbf{1}[x_j = y_j].$$

Prove the following property of P: For every $1 \leq i \leq n$, if λ_i is an eigenvalue of P_i with corresponding eigenvector v_i, then

- $v_1 \otimes v_2 \otimes \cdots \otimes v_n$ is an eigenvector of P;
- $\frac{1}{n} \sum_{i=1}^{n} \lambda_i$ is an eigenvalue of P.