ADVANCED ALGORITHMS (VII)

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1. Sparsest Cut

The problem of *sparsest cut* asks for a cut that is both *sparse* and *balanced*. For a cut $(S, \bar{S})$, define its *uniform sparsity*

$$\mathrm{usc}(S) \triangleq \frac{\left|E(S,\bar{S})\right|}{|S| \cdot \left|\bar{S}\right|}.$$

The uniform sparsest cut of a graph G is

$$\mathrm{usc}(G) \triangleq \min_S \mathrm{usc}(S).$$

In d-regular graphs, the quantity is closely related to the edge expansion $\phi(S)$ we met before. Recall that

$$\phi(S) = \frac{\left|E(S,\bar{S})\right|}{d\,|S|},$$

it is easy to see

$$\frac{\mathrm{usc}(S)}{2} \le \frac{d}{n} \cdot \phi(S) \le \mathrm{usc}(S). \tag{1}$$

Therefore, approximating $\mathrm{usc}(G)$ is equivalent to approximating $\phi(G)$ up to a factor of two. It then immediately follows from Cheeger's inequality that we can bound $\phi(G)$ in terms of the λ_2 of the normalized Laplacian N. We now show that we can directly relate λ_2 with $\mathrm{usc}(G)$.

By the variational characterization of eigenvalues, we have

$$\lambda_2 = \min_{\substack{\mathbf{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}\\ \mathbf{x}\perp\mathbf{1}}} R_N(\mathbf{x}) = \min_{\substack{\mathbf{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}\\ \mathbf{x}\perp\mathbf{1}}} \frac{\sum_{\{i,j\}\in E}(x_i - x_j)^2}{d\sum_{i\in V} x_i^2}$$

Observe that

$$\sum_{\{i,j\}\in\binom{V}{2}} = \frac{1}{2}\sum_{(i,j)\in V^2}(x_i - x_j)^2 = n\sum_{i\in V} x_i^2 - \sum_{(i,j)\in V^2} x_i x_j = n\sum_{i\in V} x_i^2 - \left(\sum_{i\in V} x_i\right)^2 = n\sum_{i\in V} x_i^2,$$

where the last equality is due to $\mathbf{x} \perp \mathbf{1}$. We can further write λ_2 as

$$\lambda_2 = \frac{n}{d} \cdot \min_{\substack{\mathbf{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}\\ \mathbf{x}\perp\mathbf{1}}} \frac{\sum_{\{i,j\}\in E}(x_i - x_j)^2}{\sum_{\{i,j\}\in\binom{V}{2}}(x_i - x_j)^2} = \frac{n}{d} \cdot \min_{\mathbf{x}\in\mathbb{R}^n\setminus\{\mathbf{0}\}} \frac{\sum_{\{i,j\}\in E}(x_i - x_j)^2}{\sum_{\{i,j\}\in\binom{V}{2}}(x_i - x_j)^2}. \tag{2}$$

In the last equality, we can safely remove the constraint $\mathbf{x} \perp \mathbf{1}$ since moving in the direction of $\mathbf{1}$ does not change the ratio we are optimizing.

On the otherhand, we can view $\mathrm{usc}(S)$ as

$$\mathrm{usc}(S) = \frac{\sum_{\{i,j\}\in E}(\mathbf{1}_S(i) - \mathbf{1}_S(j))^2}{\sum_{\{i,j\}\in\binom{V}{2}}(\mathbf{1}_S(i) - \mathbf{1}_S(j))^2},$$

which implies

$$\mathrm{usc}(G) = \min_{\mathbf{x}\in\{0,1\}^V\setminus\{\mathbf{0},\mathbf{1}\}} \frac{\sum_{\{i,j\}\in E}(x_i - x_j)^2}{\sum_{\{i,j\}\in\binom{V}{2}}(x_i - x_j)^2}. \tag{3}$$

Comparing eq. (2) and eq. (3) shows that $\frac{d}{n} \cdot \lambda_2$ is a relaxation of $\mathrm{usc}(G)$, by extending the searching range of $\mathbf{x}$ from $\{0,1\}^V \setminus \{\mathbf{0},\mathbf{1}\}$ to $\mathbb{R}^n \setminus \{\mathbf{1}\}$. This immediately implies $\frac{d}{n} \cdot \lambda_2 \le \mathrm{usc}(G)$. Along with eq. (1), we obtain an alternative proof of $\lambda_2 \le 2\phi(G)$, the lower bound part of Cheeger's inequality.

2. Leighton-Rao Relaxation

We have seen a relaxation of the uniform sparsest cut, by understanding it as an optimization problem over a hyper cube. We now introduce its non-uniform version and a better relaxation.

In the definition of uniform sparsity of S, the denominator of $\mathrm{usc}(S)$

$$\sum_{\{i,j\}\in\binom{V}{2}} (\mathbf{1}_S(i) - \mathbf{1}_S(j))^2$$

sums up all unordered pairs of vertices in V. This can be viewed as enumerating all edges of a clques whose vertex set is V. The non-uniform sparsity generalizes the clique to an arbitrary graph with vertex set V. To distinguish this graph with the original input graph, we let $G = (V, E_G)$ and $H = (V, E_H)$. Then

$$\mathrm{sc}_{G,H}(S) \triangleq \frac{\sum_{\{i,j\}\in E_G} (\mathbf{1}_S(i) - \mathbf{1}_S(j))^2}{\sum_{\{i,j\}\in E_H} (\mathbf{1}_S(i) - \mathbf{1}_S(j))^2},$$

and

$$\mathrm{sc}(G,H) \triangleq \min_{S\subseteq V} \mathrm{sc}_{G,H}(S).$$

It is clear that $\mathrm{usc}(G) = \mathrm{sc}(G, K_n)$ where $n = |V|$. We shall see in the next section a more intuitive interpretation of the non-uniformity. But for now, we first take a look at a relaxation of $\mathrm{sc}(G,H)$.

The idea is to view $(\mathbf{1}_S(i) - \mathbf{1}_S(j))^2$ as a distance between two points i and j. Specifically, if we introduce the function $d_S : V^2 \to \mathbb{R}$ as $d_S(i,j) = (\mathbf{1}_S(i) - \mathbf{1}_S(j))^2$, then it is clear that $d_S(\cdot,\cdot)$ is a semi-metric[1]. The Leighton-Rao relaxation relaxes the domain of the optimization from d_S to arbitrary semi-metric d:

$$\mathrm{LR}(G,H) \triangleq \min_{\text{semi-metric } d:V^2\to\mathbb{R}} \frac{\sum_{\{i,j\}\in E_G} d(i,j)}{\sum_{\{i,j\}\in E_H} d(i,j)}. \tag{4}$$

This optimization problem is equivalent to the following linear program, and therefore solvable in polynomial-time.

$$\begin{aligned}
\min \quad & \sum_{\{i,j\}\in E_G} d(i,j) \\
\text{s.t.} \quad & \sum_{\{i,j\}\in E_G} d(i,j) = 1 \\
& d(i,j) \ge 0, \quad \forall i,j \in V \\
& d(i,j) = d(j,i), \quad \forall i,j \in V \\
& d(i,j) + d(j,k) \ge d(i,k), \quad \forall i,j,k \in V
\end{aligned}$$

Since it is a relaxation, we have

$$\mathrm{LR}(G,H) \le \mathrm{sc}(G,H).$$

The main result today is to show that the relaxation is not too bad.

Theorem 1. *There exists a constant $c > 0$ such that*

$$\mathrm{LR}(G,H) \ge \frac{\mathrm{sc}(G,H)}{c \log n},$$

where $n = |V|$.

[1] A function $d : V^2 \to \mathbb{R}$ is a semi-metric if (1) $d(x,y) = d(y,x) \ge 0$ for every $x, y \in V$, (2) the triangle inequality holds.

3. Multicommodity Flow

Before proving theorem 1, let's see another way to come up with the expression $\mathrm{LR}(G, H)$ defined in eq. (4). The *multicommodity flow* problem is a natural generalization of the max flow problem. Suppose we are given a undirected graph $G = (V, E)$ and m pairs of sources and sinks $(s_1, t_1), \ldots, (s_m, t_m)$. Suppose each edge $\{i, j\} \in E$ has capacity 1. The problem asks for the maximum amount of flow that one can route from each source to the corresponding sink simultaniously, without violating the capacity on every edge. We use $\mathcal{P}^i = \left\{p_j^i\right\}$ to denote all simple paths between s_i and t_i. Let f be a variable indicating the max flow, or *throughput* that we want to maximized, and f_j^i be the amount of flow on each path $p_j^i \in \mathcal{P}^i$. Then the problem can be formally formulated as a linear programming:

$$
\begin{aligned}
\max \quad & f \\
\text{s.t.} \quad & \sum_{p_j^i \in \mathcal{P}^i} f_j^i \geq f, \quad \forall i = 1, \ldots, m, \\
& \sum_{p_j^i \ni e} f_j^i \leq 1, \quad \forall e \in E, \\
& f \geq 0, \\
& f_j^i \geq 0, \quad \forall i = 1, \ldots, m \text{ and each } p_j^i \in \mathcal{P}^i.
\end{aligned}
$$

The problem is the ordinary (unit capacity) max flow problem when $m = 1$. We are interested in the dual of this linear program. To this end, we introduce a dual variable ℓ_i for each $i = 1, \ldots, m$ and a dual variable y_e for each edge $e \in E$. The dual program is

$$
\begin{aligned}
\min \quad & \sum_{e \in E} y_e \\
\text{s.t.} \quad & \sum_{i=1}^{m} \ell_i \geq 1, \\
& \sum_{e \in p_j^i} y_e \geq \ell_i, \quad \forall i = 1, \ldots, m \text{ and each } p_j^i \in \mathcal{P}^i, \\
& \ell_i \geq 0, \quad \forall i = 1, \ldots, m, \\
& y_e \geq 0, \quad \forall e \in E.
\end{aligned}
$$

We write $G = (V, E_G)$ and let $H = (V, E_H)$ be a graph whose edges are those pairs of source and sink, namely $\{u, v\} \in E$ if and only if $s_i = u$ and $t_i = v$ for some $i \in [m]$. It is an exercise to verify that the dual program is equivalent to

$$
\min_{\text{semi-metric } d: V^2 \to \mathbb{R}} \frac{\sum_{\{i,j\} \in E_G} d(i, j)}{\sum_{\{i,j\} \in E_H} d(i, j)},
$$

which is exactly $\mathrm{LR}(G, H)$.

4. ℓ_1-relaxation

We introduce an "intermediate relaxation" for $\mathrm{sc}(G, H)$, namely by only allowing those ℓ_1 metrics. Define

$$
\ell_1 \mathrm{sc}(G, H) \triangleq \min_{m \geq 1} \min_{f: V \to \mathbb{R}^m} \frac{\sum_{\{i,j\} \in E_G} \|f(i) - f(j)\|_1}{\sum_{\{i,j\} \in E_H} \|f(i) - f(j)\|_1}.
$$

It is clear that $\ell_1 \mathrm{sc}(G, H)$ is a relaxation of $\mathrm{sc}(G, H)$ since $\|\mathbf{1}_S(i) - \mathbf{1}_S(j)\|_1 = (\mathbf{1}_S(i) - \mathbf{1}_S(j))^2$. Surprisingly, $\ell 1\mathrm{sc}$ does not lose anything:

Proposition 2.

$$
\ell_1 \mathrm{sc}(G, H) = \mathrm{sc}(G, H).
$$

Proof. Only need to show $\ell_1\mathrm{sc}(G,H) \geq \mathrm{sc}(G,H)$. We fix m and f that achieve the minimum in

$$\ell_1\mathrm{sc}(G,H) = \min_{m\geq 1}\min_{f:V\to\mathbb{R}^m} \frac{\sum_{\{i,j\}\in E_G}\|f(i)-f(j)\|_1}{\sum_{\{i,j\}\in E_H}\|f(i)-f(j)\|_1}.$$

Further assuming that $f(x) = (f_1(x), f_2(x), \dots, f_m(x)) \in \mathbb{R}^m$, we have

$$\begin{aligned}
\ell_1\mathrm{sc}(G,H) &= \frac{\sum_{\{i,j\}\in E_G}\|f(i)-f(j)\|_1}{\sum_{\{i,j\}\in E_H}\|f(i)-f(j)\|_1} \\
&= \frac{\sum_{\{i,j\}\in E_G}\sum_{k=1}^m \left|f_k(i)-f_k(j)\right|}{\sum_{\{i,j\}\in E_H}\sum_{k=1}^m \left|f_k(i)-f_k(j)\right|} \\
&= \frac{\sum_{k=1}^m\sum_{\{i,j\}\in E_G} \left|f_k(i)-f_k(j)\right|}{\sum_{k=1}^m\sum_{\{i,j\}\in E_H} \left|f_k(i)-f_k(j)\right|} \\
&\geq \min_k \frac{\sum_{\{i,j\}\in E_G} \left|f_k(i)-f_k(j)\right|}{\sum_{\{i,j\}\in E_H} \left|f_k(i)-f_k(j)\right|}
\end{aligned}$$

Let k^* be the one that achieves the minimum above, we only need to show

$$\frac{\sum_{\{i,j\}\in E_G} \left|f_{k^*}(i)-f_{k^*}(j)\right|}{\sum_{\{i,j\}\in E_H} \left|f_{k^*}(i)-f_{k^*}(j)\right|} \geq \mathrm{sc}(G,H),$$

or equivalent there exists some $S \subseteq V$ such that

$$\frac{\sum_{\{i,j\}\in E_G} \left|f_{k^*}(i)-f_{k^*}(j)\right|}{\sum_{\{i,j\}\in E_H} \left|f_{k^*}(i)-f_{k^*}(j)\right|} \geq \mathrm{sc}_{G,H}(S).$$

We prove the existence of S using probabilistic method. Since shifting and scaling of $f_{k^*}(\cdot)$ does not change the ratio, we can assume without loss of generality that $0 = f_{k^*}(1) \leq f_{k^*}(2) \leq \cdots \leq f_{k^*}(n) = 1$. Then we choose a real $t \in [0,1]$ uniformly at random and let $S_t = \left\{i \in V \,:\, f_{k^*}(i) \leq t\right\}$. It is easy to verify that

$$\mathbf{E}\left[\left(\mathbf{1}_{S_t}(i) - \mathbf{1}_{S_t}(j)\right)^2\right] = \left|f_{k^*}(i) - f_{k^*}(j)\right|.$$

Therefore,

$$\frac{\mathbf{E}\left[\sum_{\{i,j\}\in E_G}\left(\mathbf{1}_{S_t}(i) - \mathbf{1}_{S_t}(j)\right)^2\right]}{\mathbf{E}\left[\sum_{\{i,j\}\in E_H}\left(\mathbf{1}_{S_t}(i) - \mathbf{1}_{S_t}(j)\right)^2\right]} = \frac{\sum_{\{i,j\}\in E_G} \left|f_{k^*}(i)-f_{k^*}(j)\right|}{\sum_{\{i,j\}\in E_H} \left|f_{k^*}(i)-f_{k^*}(j)\right|}.$$

The proof can then be completed by applying the argument in the last part of the last lecture notes. □

Armed with proposition 2, we only need to compare $\mathrm{LR}(G,H)$ with $\ell_1\mathrm{sc}(G,H)$. The following powerful theorem of Jean Bourgain is at our service.

Theorem 3. *Let $d : V^2 \to \mathbb{R}$ be a semi-metric. There exists some $m \geq 1$ and a function $f : V \to \mathbb{R}^m$ such that for some constant $c > 0$ and every $x, y \in V$,*

$$\|f(x) - f(y)\|_1 \leq d(x,y) \leq c\log|V| \cdot \|f(x) - f(y)\|_1.$$

It is straightforward to prove theorem 1 using theorem 3.

Proof of theorem 1. Let d^* be the semi-metric that achieves minimum in the definition of $\mathrm{LR}(G,H)$ and $f^* : V \to \mathbb{R}$ be the one guaranteed by theorem 3 with respect to d^*. Then by proposition 2,

$$\mathrm{LR}(G,H) = \frac{\sum_{\{i,j\}\in E_G} d^*(i,j)}{\sum_{\{i,j\}\in E_H} d^*(i,j)} \geq \frac{\sum_{\{i,j\}\in E_G}\|f^*(i)-f^*(j)\|_1}{c\log n \cdot \sum_{\{i,j\}\in E_H}\|f^*(i)-f^*(j)\|_1} \geq \frac{\ell_1\mathrm{sc}(G,H)}{c\log n} = \frac{\mathrm{sc}(G,H)}{c\log n}.$$

□