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# Advanced Algorithms (II)

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We show that this ratio can also be achieved via direct **rounding**.

Recall that we have the following **linear programming relaxation**.

$$\begin{aligned} \max \quad & \sum_{j=1}^m z_j \\ \text{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{k \in N_j} (1 - y_k) \geq z_j, \quad \forall C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{k \in N_j} \bar{x}_k \\ & z_j \in [0, 1], \quad \forall j \in [m] \\ & y_i \in [0, 1], \quad \forall i \in [n] \end{aligned}$$

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For  $C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{k \in N_j} \bar{x}_k$ ,

$$\Pr [C_j \text{ is not satisfied}] = \prod_{i \in P_j} (1 - f(y_i^*)) \prod_{k \in N_j} f(y_k^*).$$



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We can choose a suitable  $f$  to get  $\frac{3}{4}$  approximation.

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The ratio  $\beta$  is called the **integrality gap** of **the** LP relaxation.

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$\mathbf{OPT} = 3$  and  $\mathbf{OPT}(LP) = 4$ .

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**Corollary.** We cannot beat  $\frac{3}{4}$  if we use  $\mathbf{OPT} \leq \mathbf{OPT}(LP)$  upper bound.

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**NP-hard**, and even hard to approximate with **any constant ratio** (unless **NP = P**).

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Oracle here: shortest  $s$ - $t$  path

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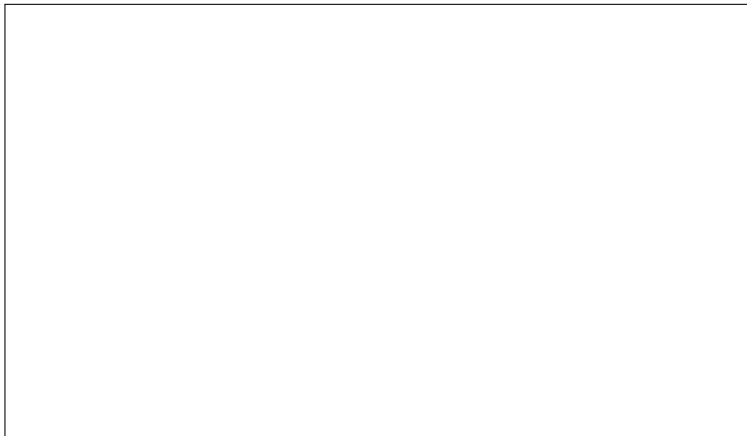
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4. Return  $L_1 \cup L_2$ .

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It is clear that

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On the other hand, there cannot be too many **edge disjoint** paths between  $s$  and  $t$  in  $G'$ :

- ▶ at least  $\frac{1}{\beta}$  edges on each  $s$ - $t$  path;
- ▶ at most  $\frac{m - |L_1|}{1/\beta} = \beta(m - |L_1|)$  such paths;
- ▶ therefore  $|L_2| \leq |F| \leq \beta(m - |L_1|)$  (**Menger's theorem**).

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Setting  $\beta = \sqrt{\frac{\mathbf{OPT}}{m}}$  yields an  $O\left(m^{\frac{1}{2}}\right)$  approximation.



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### Remark

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**Exercise.** Find an  $O\left(n^{\frac{2}{3}}\right)$ -approx algorithm via **rounding** + **BFS**.