Advanced Algorithms (II)

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Recall that we have the following linear programming relaxation.

$$\begin{aligned}
\text{max} \quad & \sum_{j=1}^{m} z_j \\
\text{subject to} \quad & \sum_{i \in P_j} y_i + \sum_{k \in N_j} (1 - y_k) \geq z_j, \quad \forall C_j = \bigvee_{i \in P_j} x_i \vee \bigvee_{k \in N_j} \bar{x}_k \\
& z_j \in [0, 1], \quad \forall j \in [m] \\
& y_i \in [0, 1], \quad \forall i \in [n]
\end{aligned}$$

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For $C_j = \bigvee_{i\in P_j} x_i \vee \bigvee_{k\in N_j} \bar{x}_k$,

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We can choose a suitable f to get $\frac{3}{4}$ approximation.

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The ratio β is called the integrality gap of the LP relaxation.

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Corollary. We cannot beat $\frac{3}{4}$ if we use $\mathbf{OPT} \leq \mathbf{OPT}(LP)$ upper bound.

Minimum Label Cut

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Minimum Label s-t Cut

Input: A graph $G = (V, E)$; a set of labels $[L] = \{1, 2, \ldots, L\}$ such that each $e \in E$ is labelled with one $\ell(e) \in [L]$; two vertices $s, t \in V$.

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NP-hard, and even hard to approximate with any constant ratio (unless $\mathbf{NP} = \mathbf{P}$).

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Oracle here: shortest s-t path

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4. Return $L_1 \cup L_2$.

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It is clear that

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On the otherhand, there cannot be too many edge disjoint paths between s and t in G':

- at least $\frac{1}{\beta}$ edges on each s-t path;
- at most $\frac{m-|L_1|}{1/\beta} = \beta(m - |L_1|)$ such paths;
- therefore $|L_2| \leq |F| \leq \beta(m - |L_1|)$ (Menger's theorem).

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Exercise. Find an $O\left(n^{\frac{2}{3}}\right)$-approx algorithm via rounding + BFS.