Advanced Algorithms (III)

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MaxCut

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Problem: A set $S \subseteq V$ that maximizes $\left|E(S, \overline{S})\right|$.

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Can we find clever coins via LP relaxation...?

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idea: Let $F = \left\{\{u,v\} \in E \,:\, y_{u,v} = 1\right\}$, we view $(S, \bar{S}, F)$ as a bipartite subgraph of G.

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\text{s.t.} \quad & y_{u,v} \le y_{u,w} + y_{w,v}, \quad \forall u,v,w \in V
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& \sum_{e=\{u,v\}\in C} y_{u,v} \le |C| - 1, \quad \forall \text{odd cycle } C
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Theorem

For every $\varepsilon > 0$, there exists a graph G such that

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Random graph $\mathcal{G}(n, p)$ for proper p...

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\max \quad & 2x - 3y \\
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Hadamard Product

$$A \bullet B \triangleq \sum_{1 \le i,j \le n} a_{ij} \cdot b_{ij}.$$

Positive Semi-definite Matrix

Definition

An $n \times n$ symmetric matrix A is positive semi-definite if $x^T A x \geq 0$ for every vector x. We write it as

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Linear Programming

$$\begin{aligned} \max \quad & c^T x \\ \text{s.t.} \quad & a_i^T x \leq b_i, \quad \forall i \in [m] \\ & x_j \geq 0, \quad \forall j \in [n] \end{aligned}$$

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PSD Programming

$$\begin{aligned} \max \quad & C \bullet X \\ \text{s.t.} \quad & A_i \bullet X \leq b_i, \quad \forall i \in [m] \\ & X \succeq 0 \end{aligned}$$

Property of PSD Matrix

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Theorem

For an $n \times n$ symmetric matrix, the followings are equivalent

1. $A \succeq 0$;
2. A has n non-negative eigenvalues;
3. $A = V^T V$ for some $n \times n$ matrix $V = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$.

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We now prove the theorem using spectral theorem for symmetric matrices.

Vector Programming

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PSD Programming

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Vector Programming

$$\begin{aligned} \max \quad & \sum_{1 \le i,j \le n} c(i,j) \cdot \mathbf{v}_i^T \mathbf{v}_j \\ \text{s.t.} \quad & \sum_{1 \le i,j \le n} a_k(i,j) \cdot \mathbf{v}_i^T \mathbf{v}_j \le b_k, \quad \forall k \in [m] \\ & \mathbf{v}_i \in \mathbb{R}^n, \quad \forall i \in [n] \end{aligned}$$

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Rounding

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Next week: How to implement the rounding? How to analyze the performance?