Advanced Algorithms (IV)

Chihao Zhang

Shanghai Jiao Tong University

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Review

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MaxCut

Input: An undirected graph $G = (V, E)$.

Problem: A set $S \subseteq V$ that maximizes $\left|E(S, \overline{S})\right|$.

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Vector Program

$$\begin{aligned} \max \quad & \frac{1}{2} \sum_{e=\{u,v\}\in E} \left(1 - \mathbf{w}_u^T \mathbf{w}_v\right) \\ \text{s.t.} \quad & \mathbf{w}_u \in \mathbb{R}^n, \quad \forall u \in V \\ & \mathbf{w}_u^T \mathbf{w}_u = 1, \quad \forall u \in V \end{aligned}$$

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Let $\{\widehat{\mathbf{w}_v}\}_{v \in V}$ be an optimal solution.

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Task: Round $\{\widehat{\mathbf{w}_v}\}_{v\in V}$ to a cut

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Implementation

1. Choose a vector $\mathbf{r} = (r_1, \ldots, r_n)$ where each $r_i \sim \mathcal{N}(0, 1)$ i.i.d.
2. Let $S \triangleq \left\{u \in V : \mathbf{r}^T \widehat{\mathbf{w}_u} \geq 0\right\}$.

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Proposition

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Random hyperplane rounding is a 0.878-approximation of MaxCut.

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We assume $A = (a_{i,j})_{1 \le i,j \le n}$ is positive semi-definite.

Rounding

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Vector Program

$$\begin{aligned} \max \quad & \sum_{1\le i,j\le n} a_{i,j}\mathbf{v}_i^T\mathbf{v}_j \\ \text{s.t.} \quad & \mathbf{v}_i \in \mathbb{R}^n, \quad i = 1,\ldots,n. \end{aligned}$$

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$$\max \quad \sum_{1\le i,j\le n} a_{i,j}\mathbf{v}_i^T\mathbf{v}_j$$
$$\text{s.t.} \quad \mathbf{v}_i \in \mathbb{R}^n, \quad i = 1, \ldots, n.$$

1. Compute $\{\widehat{\mathbf{v}}_i\}_{1\le i\le n}$.
2. Pick a vector $\mathbf{r}$ u.a.r on S^{n-1}.
3. $\hat{x}_i = 1$ if ${\widehat{\mathbf{v}}_i}^T\mathbf{r} \ge 0$; $\hat{x}_i = -1$ otherwise.

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Proof.

Use Schur producet theorem. □

Correlation Clustering

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- Find a partition $\mathcal{S} = (S_1, \ldots, S_k)$ of V.
- $E^+(\mathcal{S}) \triangleq$ edge in a cluster; $E^-(\mathcal{S}) \triangleq$ edges between clusters.
- The goal is to maximize

$$\sum_{e \in E^+(\mathcal{S})} w_e^+ + \sum_{e \in E^-(\mathcal{S})} w_e^- .$$

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Relaxation

$$\begin{aligned} \max \quad & \sum_{\{u,v\}\in E} \left(w^+_{u,v}(x_u^T x_v) + w^-_{u,v}(1 - x_u^T x_v) \right) \\ \text{s.t.} \quad & x_v^T x_v = 1, \quad \forall v \in V, \\ & x_u^T x_v \geq 0, \quad \forall u, v \in V, \\ & x_u \in \mathbb{R}^n, \quad \forall u \in V. \end{aligned}$$

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Proposition

Two random hyperplane rounding is a $\frac{3}{4}$-approximation for correlation clustering.