Advanced Algorithms (X)

Shanghai Jiao Tong University

Chihao Zhang

May 11, 2020

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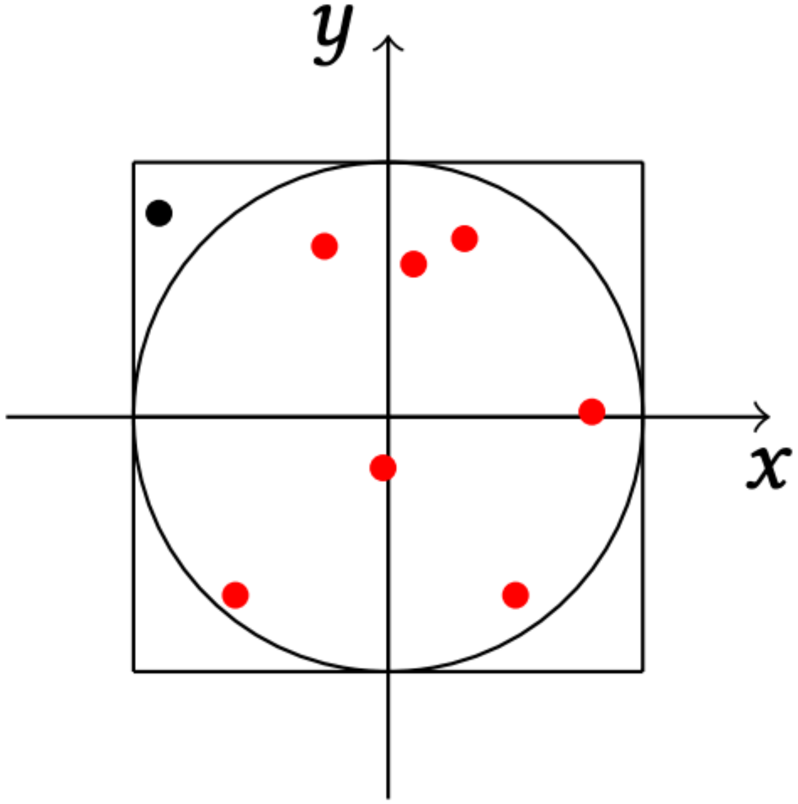

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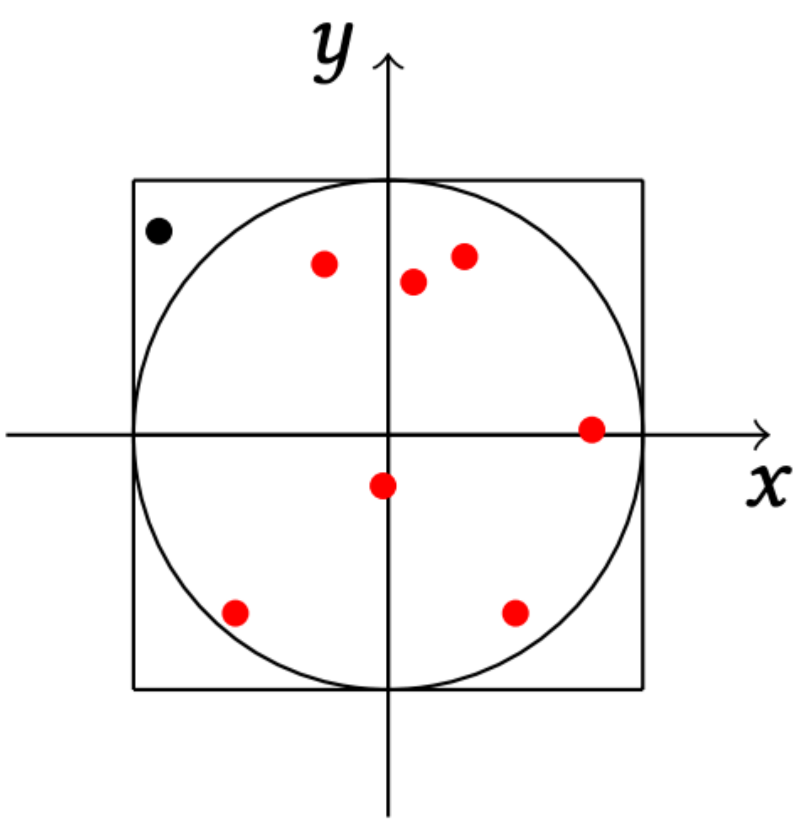

- $X_i \in [-1,1] \times [-1,1]$
- $Z_n = \sum_{i=1}^{n} \mathbf{1}[\|X_i\| \leq 1]$

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If $n \geq \dfrac{12}{\varepsilon^2 \pi} \log \dfrac{2}{\delta}$, we have an $1 \pm \varepsilon$ approximation of π with probability at least $1 - \delta$

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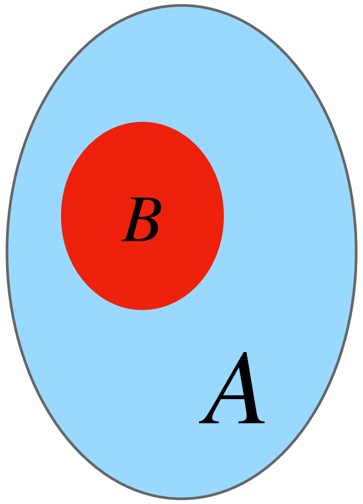

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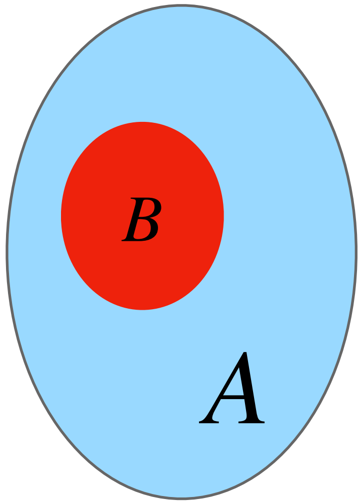

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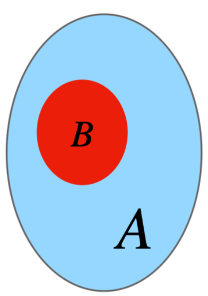

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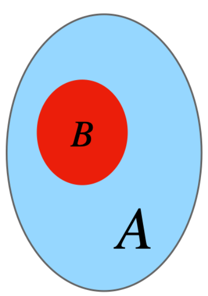

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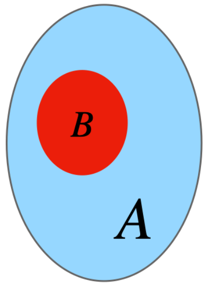

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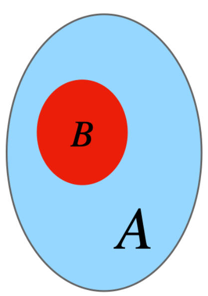

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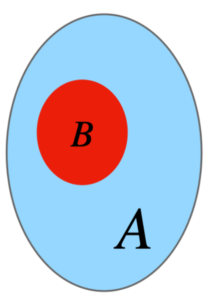

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The Monte Carlo method using rejection sampling is slow!

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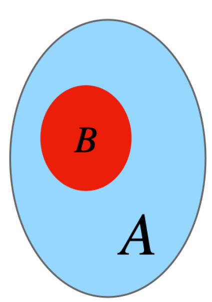

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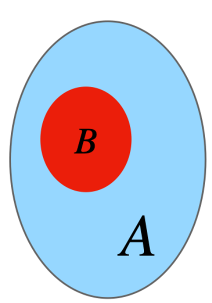

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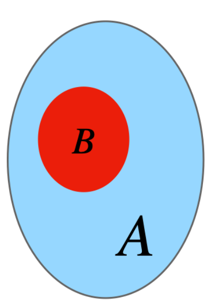

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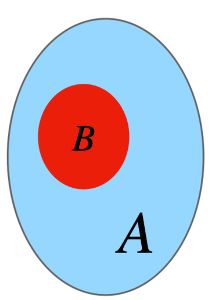

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(disjoint union)

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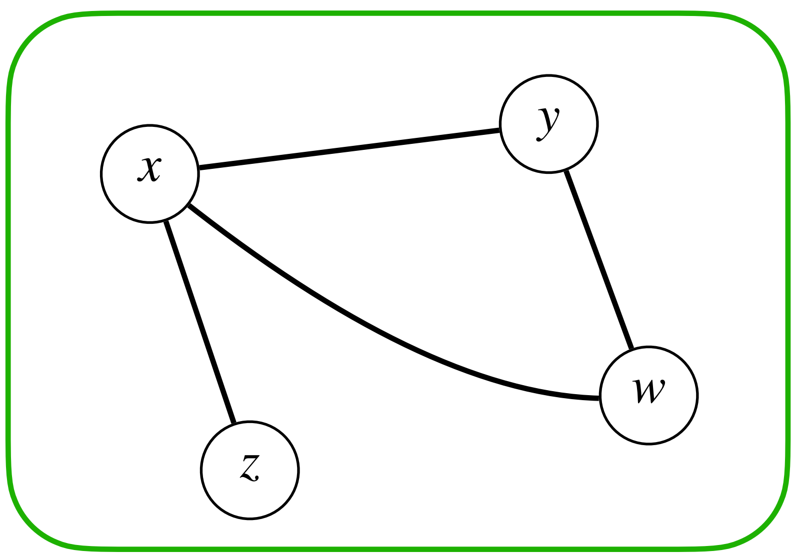

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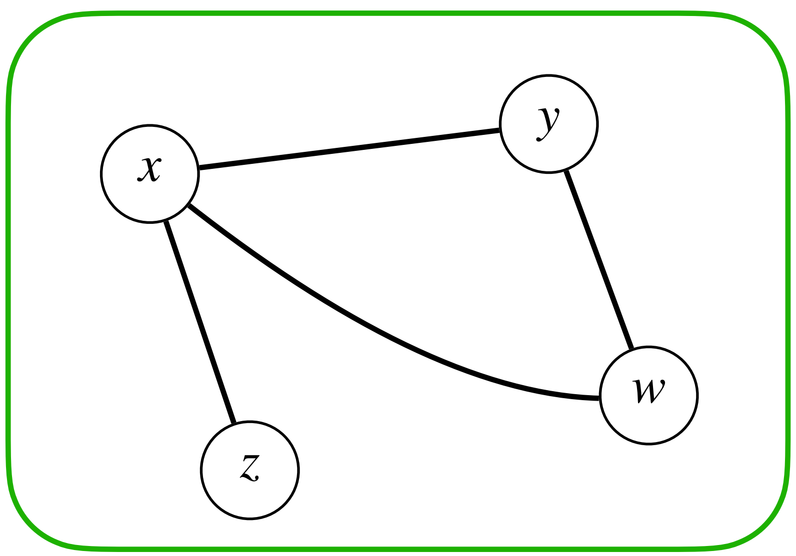
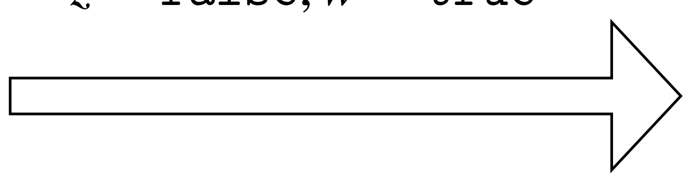


$x = \texttt{true}, y = \texttt{false}$

$z = \texttt{false}, w = \texttt{true}$



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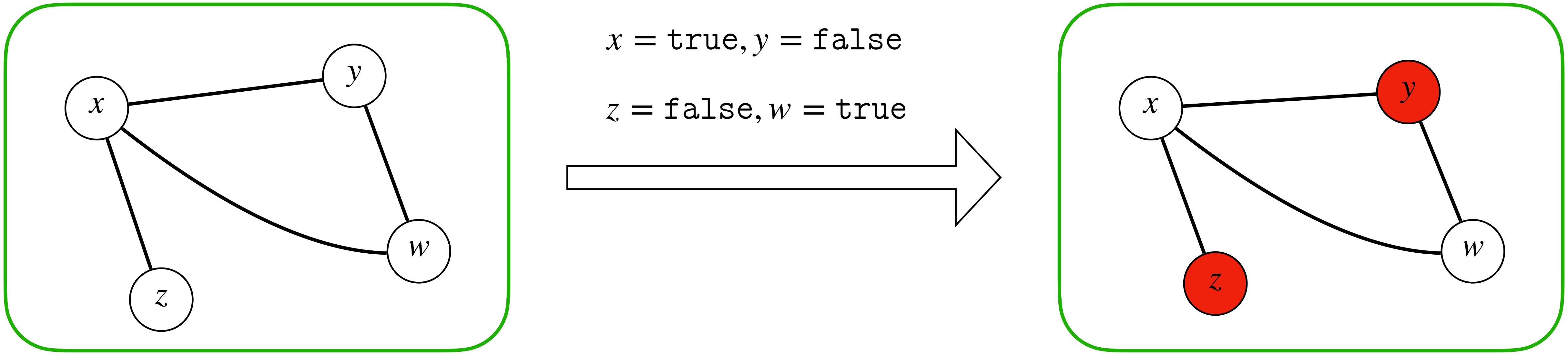

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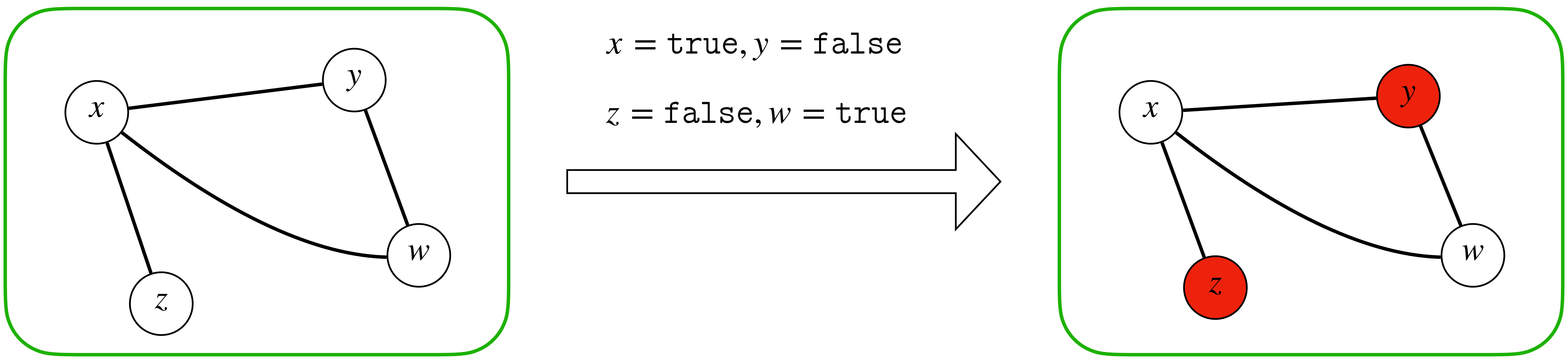

$$\#\varphi = \#\text{ of independent sets}$$

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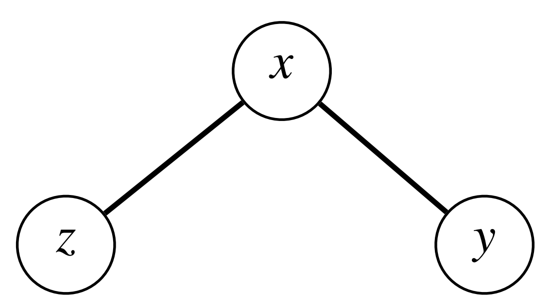
Think about 

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We will prove the correctness and analyze its efficiency next week

From Sampling to Counting

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We want to estimate $I(G)$, the number of i.s. in G

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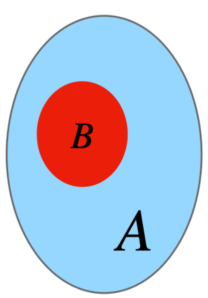

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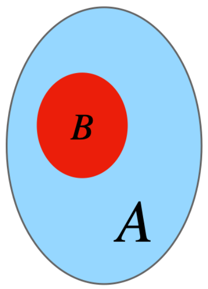

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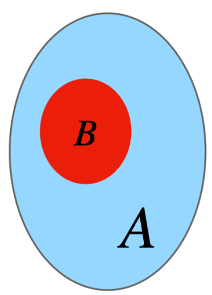

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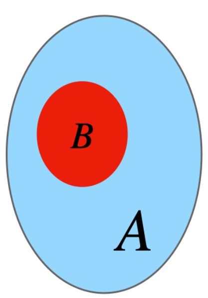

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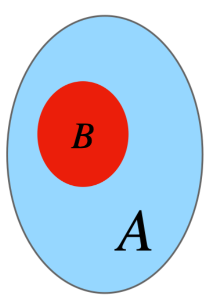

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$$\frac{I(G_i)}{I(G_{i+1})} \leq 2$$

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The above two reductions require the system to satisfy "self-reducible" property