Advanced Algorithms (XIV)

Shanghai Jiao Tong University

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Mixing Time via Coupling

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The state space Ω

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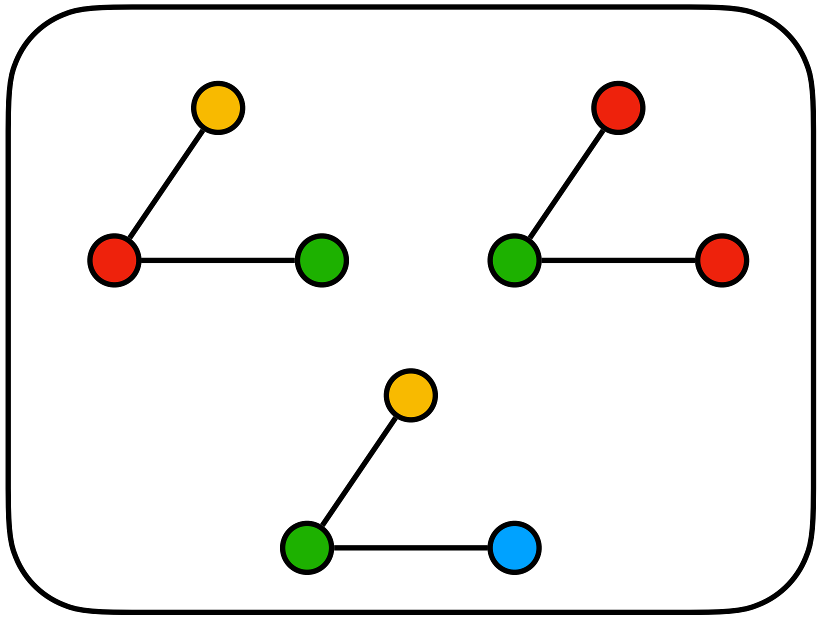

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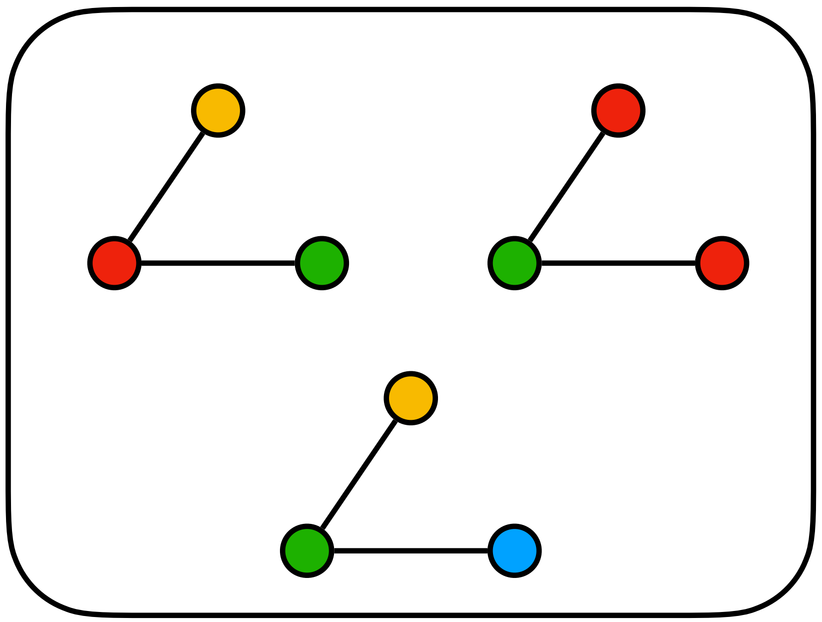

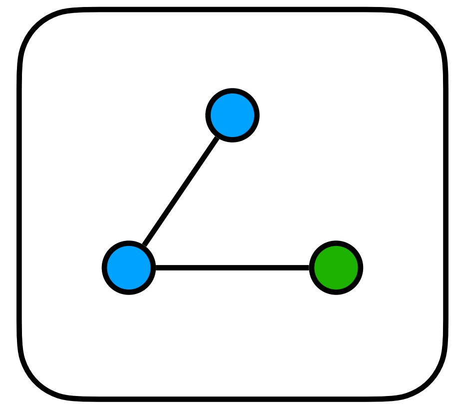

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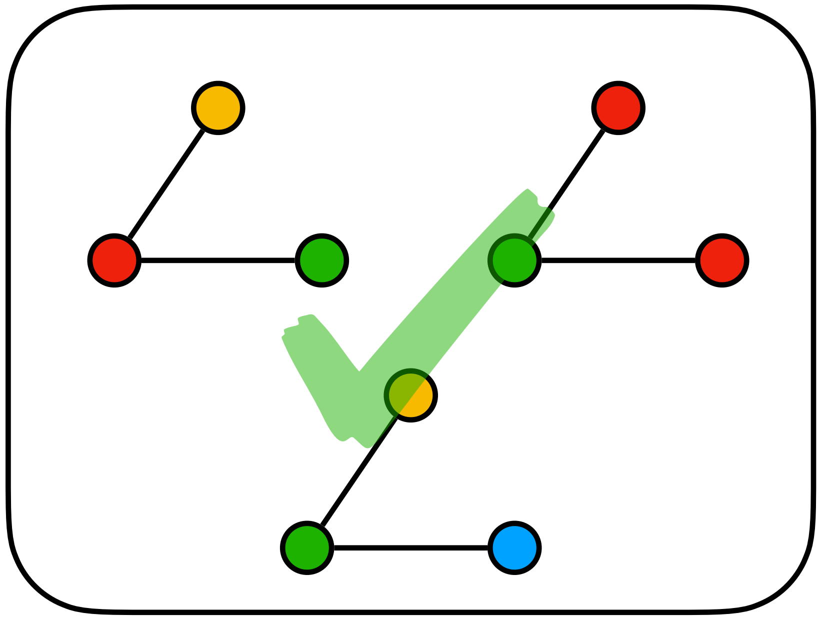

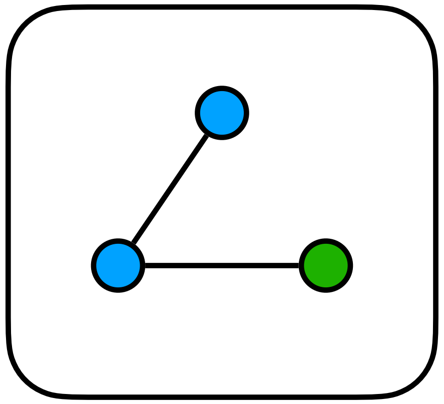

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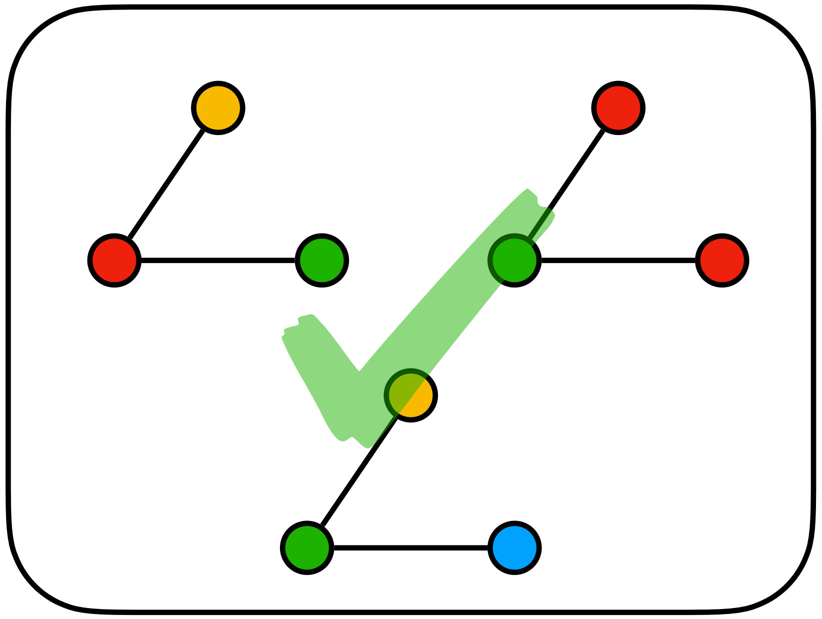

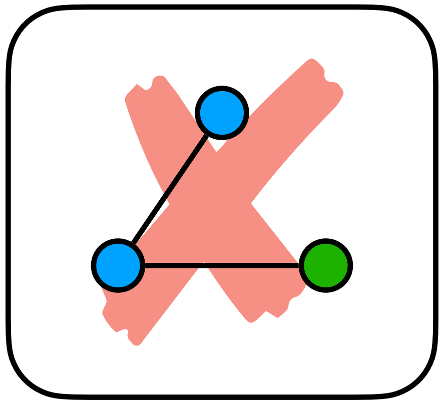

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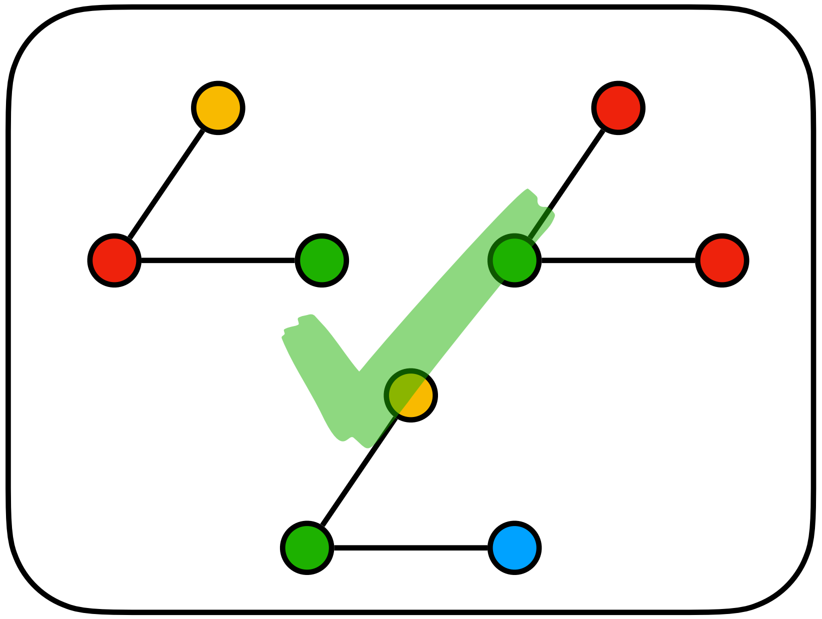

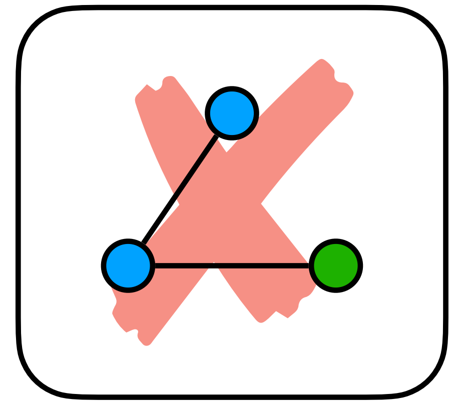

q - the number of proper colorings

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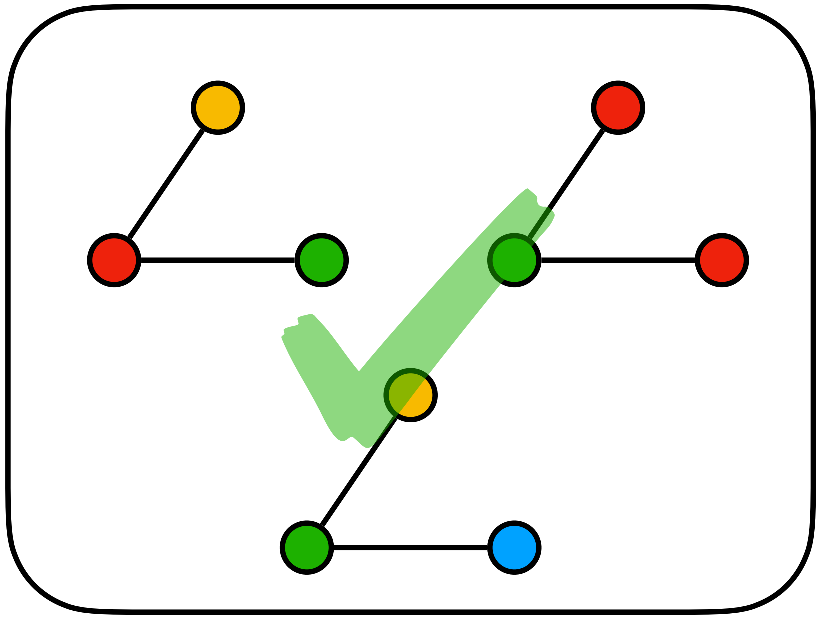

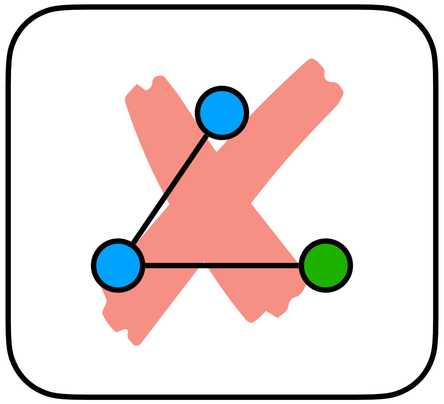

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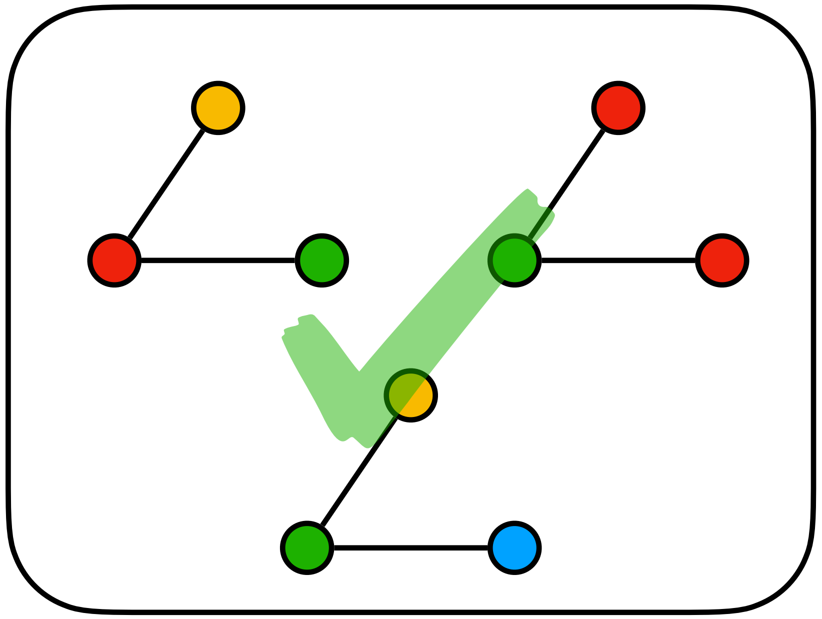

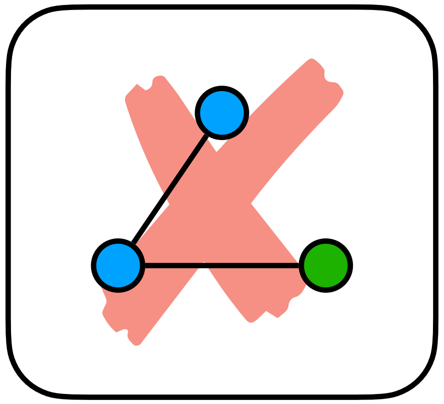

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Is G colorable using q colors?

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The chain is irreducible when $q \geq \Delta + 2$

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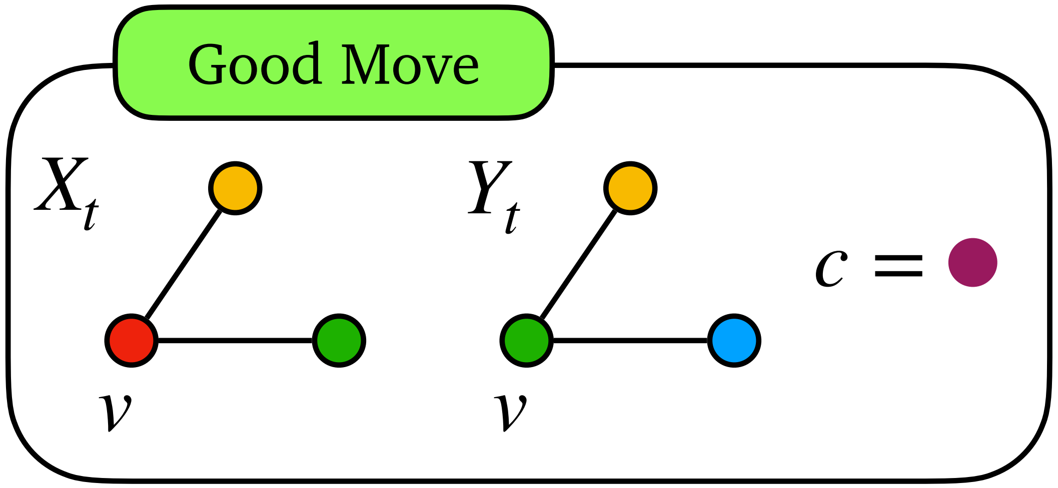

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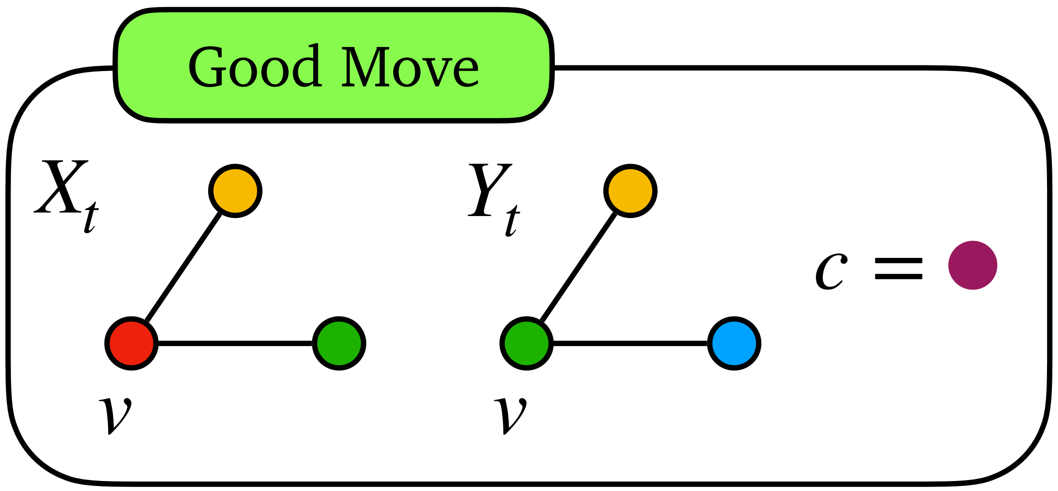

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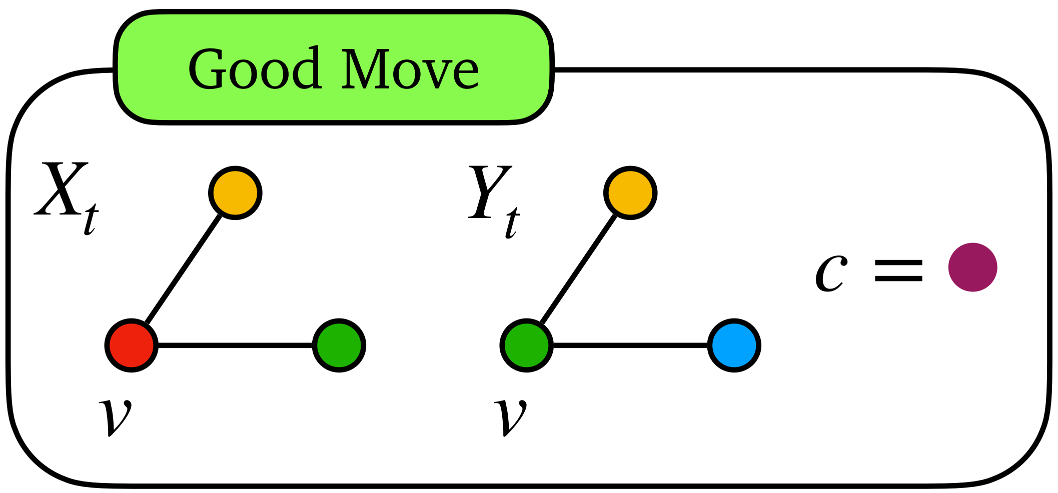

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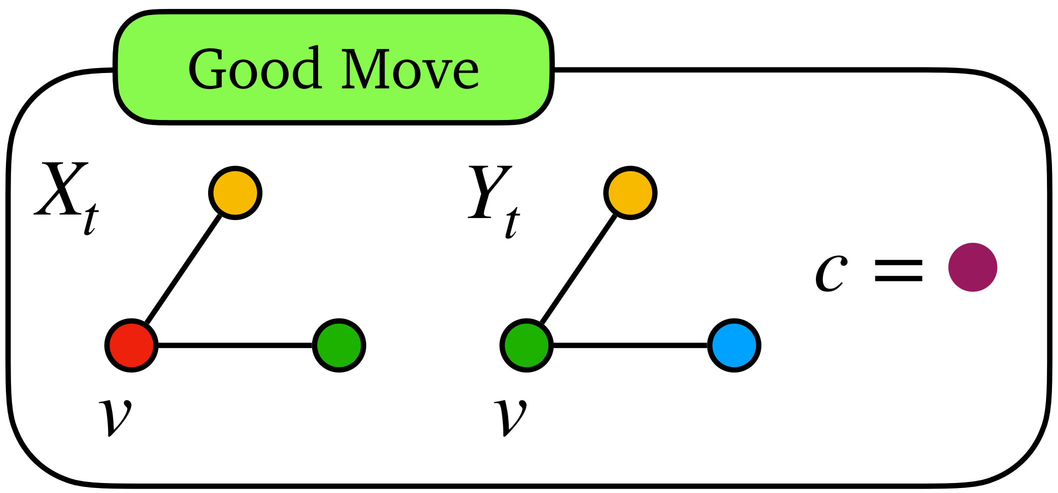

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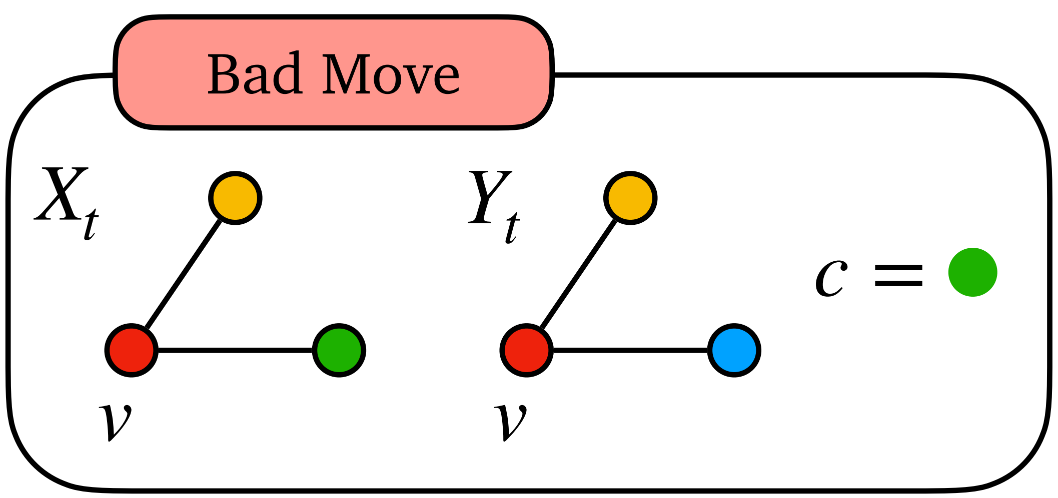

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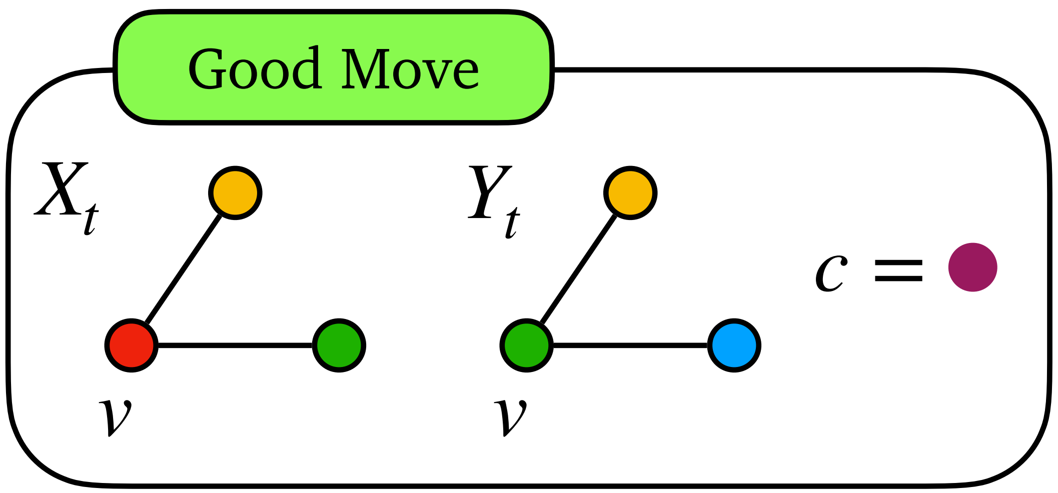

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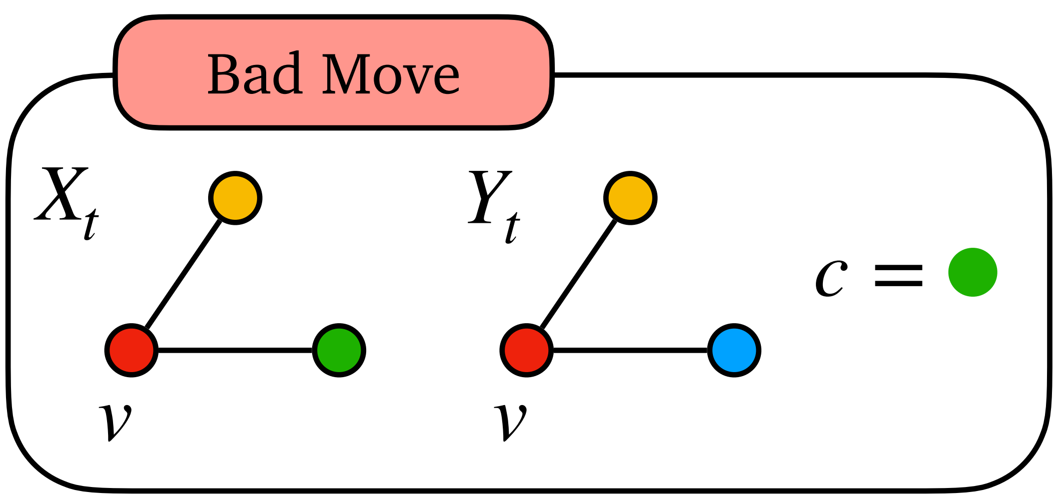

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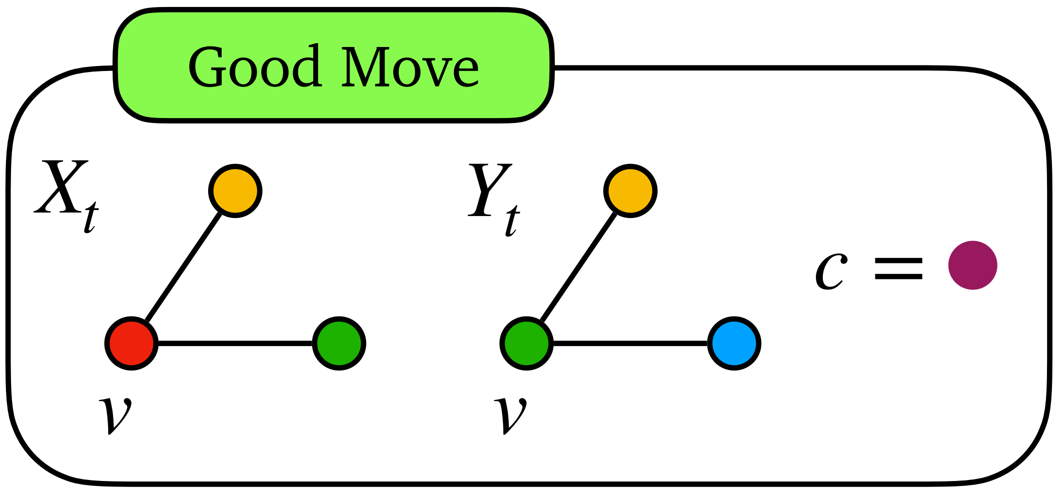

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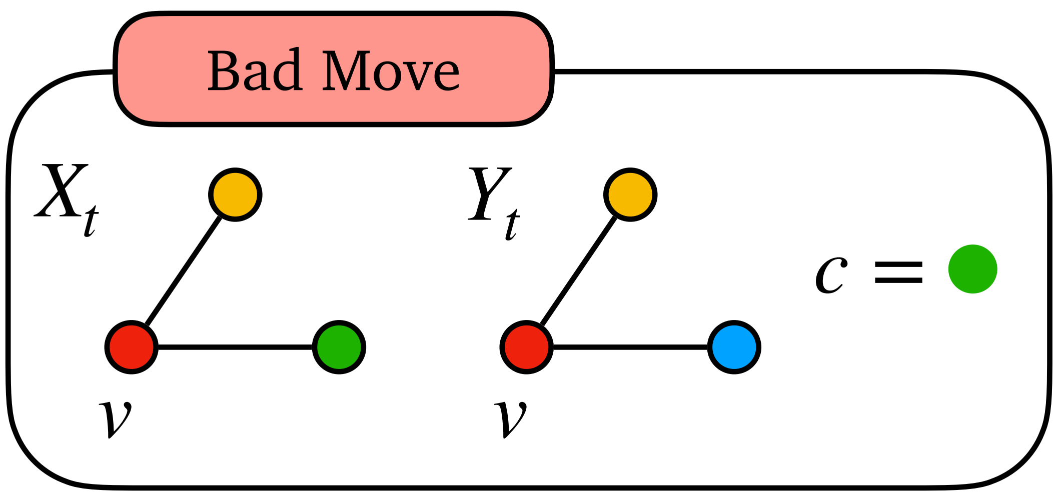

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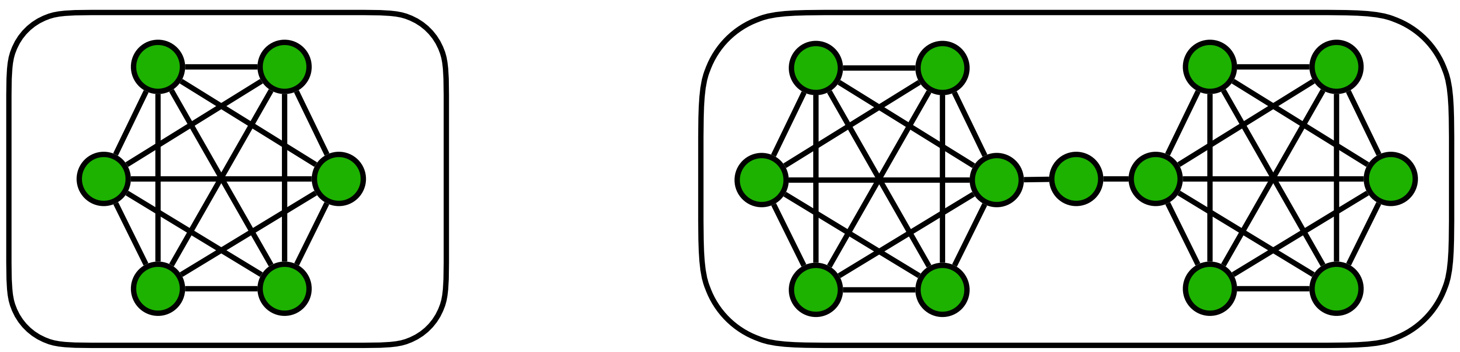

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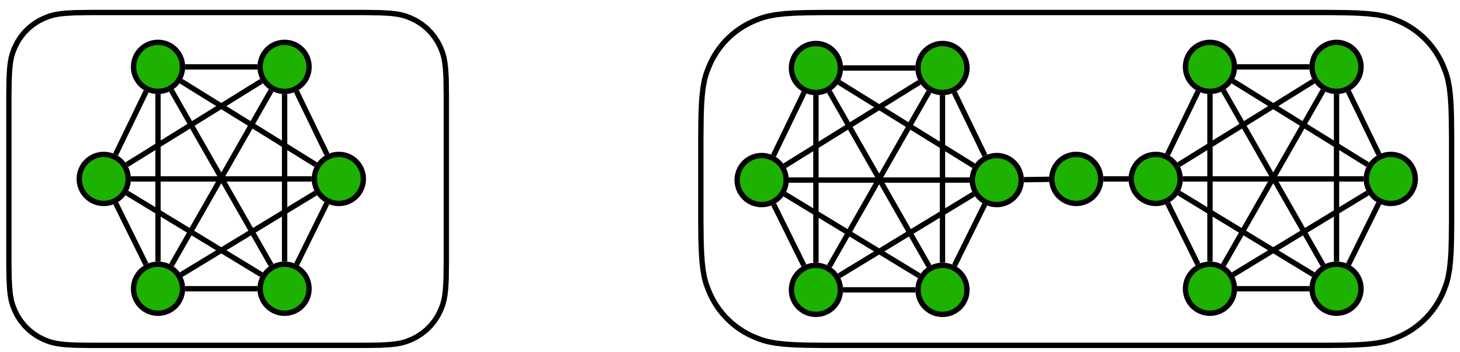

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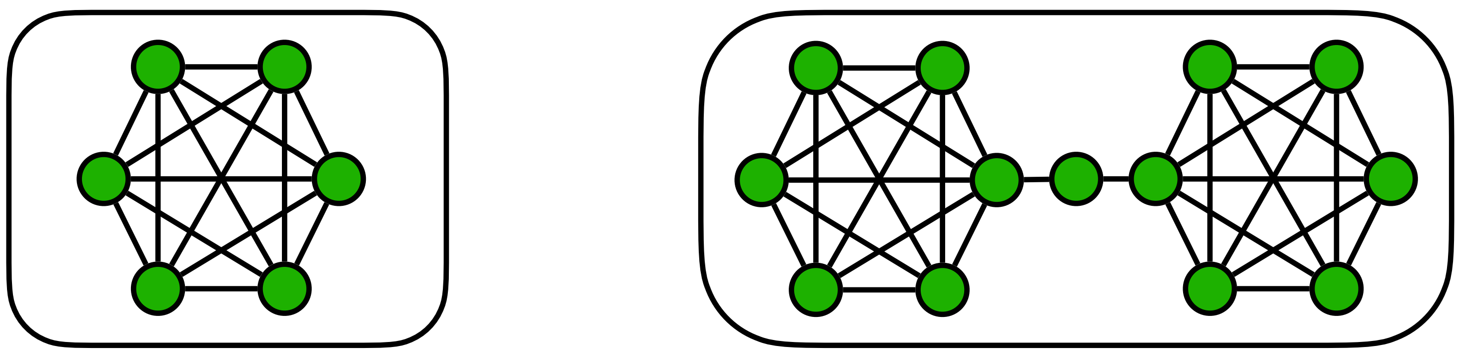

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We will develop tools to formalize the intuition

Back to Graph Spectrum