Advanced Algorithms (VI)

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Martingale

Let $\{X_t\}_{t \geq 0}$ be a sequence of random variables

Let $\{\mathcal{F}_t\}_{t \geq 0}$ be a sequence of σ-algebras such that

$$\mathcal{F}_0 \subseteq \mathcal{F}_1 \subseteq \mathcal{F}_2 \cdots$$

filtration

A martingale is a sequence of pairs $\{X_t, \mathcal{F}_t\}_{t \geq 0}$ s.t.

- for all $t \geq 0$, X_t is $\mathcal{F}_t$-measurable
- for all $t \geq 0$, $\mathbf{E}[X_{t+1} \mid \mathcal{F}_t] = X_t$

Stopping Time

The stopping time $\tau \in \mathbb{N} \cup \{\infty\}$ is a random variable such that

$$[\tau \leq t] \text{ is } \mathscr{F}_t\text{-measurable for all } t$$

"whether to stop can be determined by looking at the outcomes seen so far"

- The first time a gambler wins five games in a row
- The last time a gambler wins five games in a row

A basic property of a martingale $\{X_t, \mathscr{F}_t\}_{t \geq 0}$ is $\mathbf{E}[X_t] = \mathbf{E}[X_0]$ for any $t \geq 0$

Proof. $\forall t \geq 1, \mathbf{E}[X_t] = \mathbf{E}[\mathbf{E}[X_t \mid \mathscr{F}_{t-1}]] = \mathbf{E}[X_{t-1}]$

Does $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ hold for a (randomized) stopping time τ?

Not true in general. Assume τ is the first time a gambler wins \$100

Optional Stopping Theorem

For a stopping time τ, $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ holds if

- $\Pr[\tau < \infty] = 1$
- $\mathbf{E}[\,|X_\tau|\,] < \infty$
- $\lim_{t\to\infty} \mathbf{E}[X_t \cdot \mathbf{1}_{[\tau>t]}] = 0$

The following conditions are stronger, but easier to verify

1. There is a fixed n such that $\tau \leq n$ a.s.

2. $\Pr[\tau < \infty] = 1$ and there is a fixed M such that $|X_t| \leq M$ for all $t \leq \tau$

3. $\mathbf{E}[\tau] < \infty$ and there is a fixed c such that $|X_{t+1} - X_t| \leq c$ for all $t < \tau$

OST applies when at least one of above holds

Proof of the Optional Stopping Theorem

Applications of OST

Random Walk in 1-D

Let $Z_t \in \{-1, +1\}$ u.a.r. and $X_t = \sum_{i=1}^{t} Z_i$

The random walk stops when it hits $-a < 0$ or $b > 0$

Let τ be the time it stops. τ is a stopping time

What is $\mathbf{E}[\tau]$?

The random walk stops when one of two ends is arrived

We first determine p_a, the probability that the walk ends at $-a$, using OST

$$\begin{aligned} \mathbf{E}[X_\tau] &= p_a(-a) + (1 - p_a)b \\ &= \mathbf{E}[X_0] = 0 \end{aligned}$$

$$\implies p_a = \frac{b}{a+b}$$

Conditions for OST

1. There is a fixed n such that $\tau \le n$ a.s.
2. $\Pr[\tau < \infty] = 1$ and there is a fixed M such that $|X_t| \le M$ for all $t \le \tau$
3. $\mathbf{E}[\tau] < \infty$ and there is a fixed c such that $|X_{t+1} - X_t| \le c$ for all $t < \tau$

Now define a random variable $Y_t = X_t^2 - t$

Claim. $\{Y_t\}_{t \geq 0}$ is a martingale

$$
\begin{aligned}
\mathbf{E}[Y_{t+1} \mid \mathcal{F}_t] &= \mathbf{E}[(X_t + Z_{t+1})^2 - (t+1) \mid \mathcal{F}_t] \\
&= \mathbf{E}[X_t^2 + 2Z_{t+1}X_t - t \mid \mathcal{F}_t] \\
&= X_t^2 - t = Y_t
\end{aligned}
$$

Y_τ satisfies the condition for OST, so

$$\mathbf{E}[Y_\tau] = \mathbf{E}[X_\tau^2] - \mathbf{E}[\tau] = \mathbf{E}[Y_0] = 0$$

Conditions for OST

1. There is a fixed n such that $\tau \le n$ a.s.
2. $\Pr[\tau < \infty] = 1$ and there is a fixed M such that $|X_t| \le M$ for all $t \le \tau$
3. $\mathbf{E}[\tau] < \infty$ and there is a fixed c such that $|X_{t+1} - X_t| \le c$ for all $t < \tau$

On the other hand, we have

$$\mathbf{E}[X_\tau^2] = p_a \cdot a^2 + (1 - p_a) \cdot b^2 = ab$$

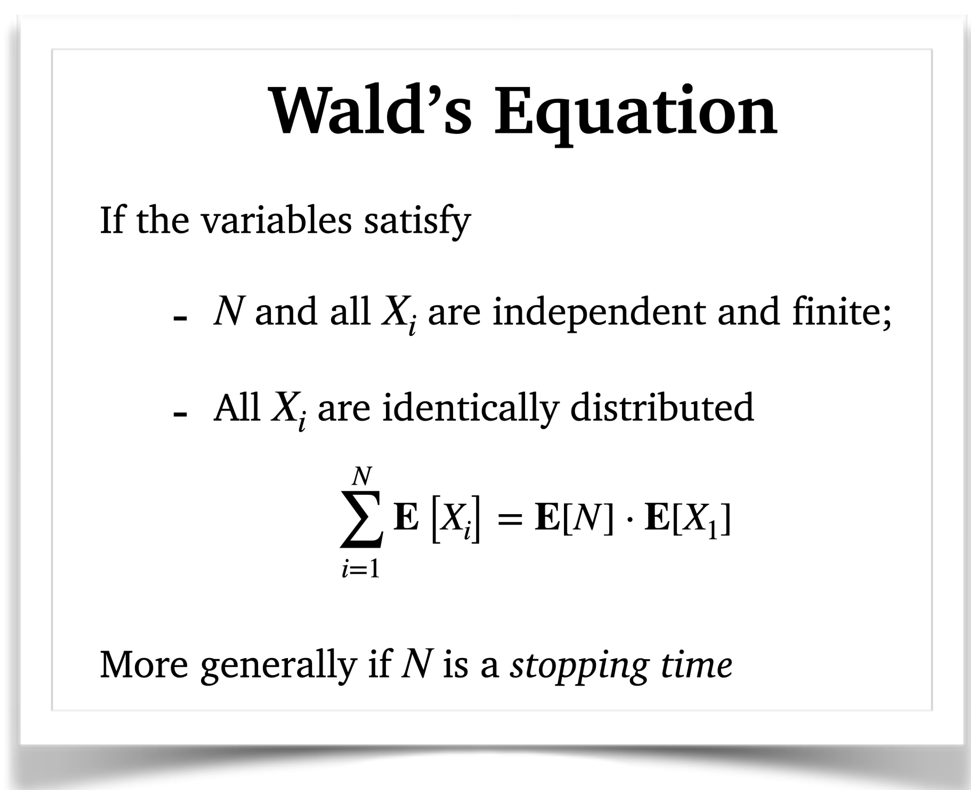
This implies $\mathbf{E}[\tau] = ab$

Wald's Equation

Recall in Week two, we consider the sum $\mathbf{E}\left[\sum_{i}^{N} X_i\right]$

where $\{X_i\}$ are independent with mean μ and N is a random variable

We are now ready to prove the general case!



Wald's Equation

If the variables satisfy

- N and all X_i are independent and finite;
- All X_i are identically distributed

$$\sum_{i=1}^{N} \mathbf{E}\left[X_i\right] = \mathbf{E}[N] \cdot \mathbf{E}[X_1]$$

More generally if N is a *stopping time*

Assume $\mathbf{E}[N]$ is finite and let $Y_t = \sum_{i=1}^{t} (X_i - \mu)$

$\{Y_t\}$ is a martingale and the stopping time N satisfies the conditions for OST

$$\mathbf{E}[Y_N] = \mathbf{E}\left[\sum_{i=1}^{N} (X_i - \mu)\right] = \mathbf{E}\left[\sum_{i=1}^{N} X_i\right] - \mathbf{E}\left[\sum_{i=1}^{N} \mu\right]$$

$$= \mathbf{E}\left[\sum_{i=1}^{N} X_i\right] - \mathbf{E}[N] \cdot \mu = 0$$

Waiting Time for Patterns

Fix a pattern $P =$ "00110"

How many fair coins one needs to toss to see P for the first time (in expectation)?

Shuo-Yen Robert Li (李碩彥)

The number can be calculated using OST

Let the pattern $P = p_1 p_2 \ldots p_k$

We draw a random string $B = b_1 b_2 b_3 \ldots$

Imagine for each $j \geq 1$, there is a gambler G_j

At time j, G_j bets \$1 for "$b_j = p_1$". If he wins, he bets \$2 for "$b_{j+1} = p_2$", …

He keeps doubling the money until he loses

The money of G_j is a martingale (w.r.t. B)

Let X_t be the money of all gamblers at time t

$\{X_t\}_{t\geq 1}$ is also a martingale

Let τ be the first time that we meet P in B

$\{X_t\}$ and τ meet the conditions for OST, so $\mathbf{E}[X_\tau] = 0$

Now we can compute the money of each G_j at τ

- All gamblers before $\tau - k + 1$ must lose
- The gambler $G_{\tau-k+1}$ wins $2^k - 1$
- Any other gamblers can win?

A gambler $G_{\tau-j+1}$ wins iff $p_1 p_2 \ldots p_j = p_{k-j+1} p_{k-j+2} \ldots p_k$

If $G_{\tau-j+1}$ wins, he wins $\$2^j - 1$

For any $P = p_1 p_2 \ldots p_k$ and $1 \le j \le k$, let χ_j be the indicator that $p_1 \ldots p_j = p_{k-j+1} \ldots p_k$

Then $X_\tau = -\left(\tau - \sum_{j=1}^{k} \chi_j\right) + \sum_{j=1}^{k} \chi_j \cdot (2^j - 1)$

contribution of losers

contribution of winners

This implies $\mathbf{E}[\tau] = \sum_{j=1}^{k} \chi_j \cdot 2^j$

Proof of OST

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Read Chapter 8 of "Notes on Randomized Algorithms" for more details

https://arxiv.org/abs/2003.01902