Advanced Algorithms (VIII)

Shanghai Jiao Tong University

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The Probabilistic Method

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Design a probability space Ω

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Show that $\Pr[\text{the object exists}] > 0$

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Is $\Pr[\bar{A}_1 \wedge \bar{A}_2 \ldots \wedge \bar{A}_m] > 0$?

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The union bound is tight when bad events are disjoint

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The two cases correspond to two extremes of the dependency

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Erdős and Lovász, *Infinite and Finite Sets*, 1975

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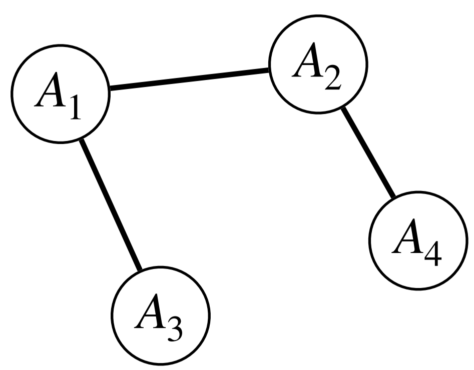

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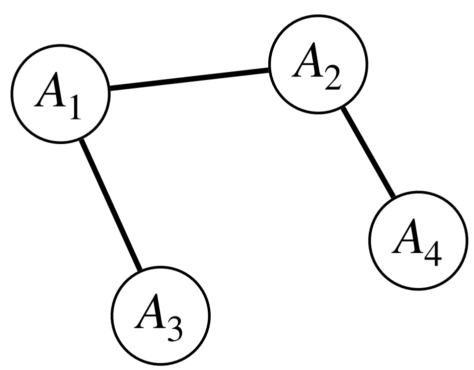

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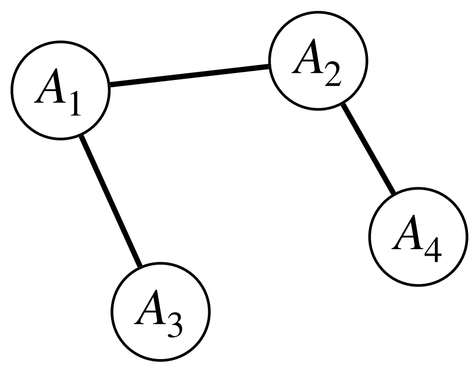

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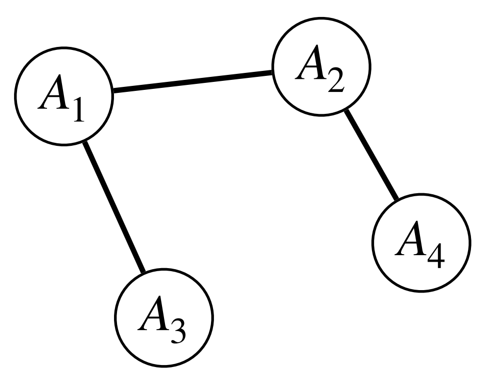

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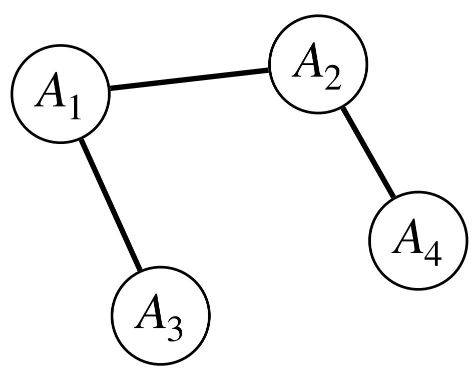

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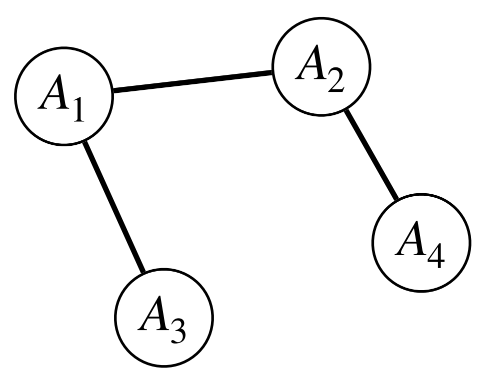

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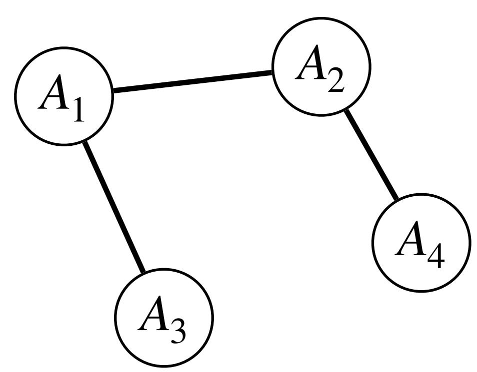

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Proof of (Symmetric) LLL

For $S \subseteq [m]$, we prove by induction on $|S|$ that

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For every $T \subseteq [m]$, we use F_T to denote the event $\bigcap_{i \in T} \bar{A}_i$

It is clear that for every $T \in \binom{[m]}{\leq s}$,

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If $|S_2| = s$, then $\Pr[A_i \mid S] = \Pr[A_i \mid S_2] \leq p$

Otherwise,

$$\Pr[A_i \mid F_S] = \Pr[A_i \mid F_{S_1} \cap F_{S_2}] = \frac{\Pr[A_i \cap F_{S_1} \cap F_{S_2}]}{\Pr[F_{S_1} \cap F_{S_2}]}$$

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$$\Pr[F_{S_1} \mid F_{S_2}] = 1 - \Pr\left[\bigcup_{j \in S_1} A_j \mid F_{S_2}\right] \geq 1 - 2dp \geq \frac{1}{2}$$

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$$\implies \quad \Pr[A_i \mid F_S] \le 2p$$

Applications of LLL

Edge-Disjoint Paths

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If $8nk \leq m$, then there is a way to choose n edge-disjoint paths connecting n pairs

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So we only need to show $\Pr\left[\bigcap_{\{i,j\}\in\binom{n}{2}} \bar{E}_{ij}\right] > 0$

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The LLL condition is then $8nk \leq m$

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Let d be the maximum degree of variables in ϕ

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Then $\Pr\left[\bigcap_{i=1}^{n} \bar{A}_i\right] \ge \prod_{i=1}^{n} (1 - x_i)$

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The Gödel Prize 2020 - Laudation

The 2020 Gödel Prize is awarded to **Robin A. Moser** and **Gábor Tardos** for their algorithmic version of the Lovász Local Lemma in the paper:

"A constructive proof of the general Lovász Local Lemma," Journal of the ACM 57(2): 11:1-11:15 (2010).

The Lovász Local Lemma (LLL) is a fundamental tool of the probabilistic method. It enables one to show the existence of certain objects even though they occur with exponentially small probability. The original proof was not algorithmic, and subsequent algorithmic versions had significant losses in parameters. This paper provides a simple,