Algorithms for Big Data (XIII)

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Review

We studied random walks on general graphs using spectral decomposition.

We introduced the notion of electrical networks.

We derived bounds on the cover time of random walks.

Electrical Network

Now we formally justify the electrical network argument used last week.

For an edge with weight w_e, we define its resistance $r_e = w_e^{-1}$.

For an edge $\{u, v\}$, we can assign numbers $\mathbf{i}(u, v) = -\mathbf{i}(v, u)$ as the current on the edge.

The collection of currents is required to satisfy Kirchhoff's law.

Ohm's law is used to define the potential drop between two ends of an edge.

Matrix Form

It is instructive to express physical laws in the matrix form.

We use an ordered pair (u, v) satisfying $u \leq v$ to represent an edge $\{u, v\} \in E$.

The signed edge-vertex adjacency matrix $U \in \{0, 1, -1\}^{E \times V}$ is defined as

$$U((u, v), w) = \begin{cases} 1 & \text{if } w = u \\ -1 & \text{if } w = v \\ 0 & \text{otherwise.} \end{cases}$$

Let $W \in \mathbb{R}^{E \times E}$ be $\mathrm{diag}(w(e_1), \ldots, w(e_{|E|}))$.

We use $\mathbf{i} \in \mathbb{R}^E$ to denote the vector of currents, $\mathbf{v} \in \mathbb{R}^V$ to denote the vector of voltages.

It holds that

$$\mathbf{i} = W \cdot U \cdot \mathbf{v}.$$

We use $\mathbf{i}_{\text{ext}}(u)$ to denote the amount of current entering u externally.

Then $\mathbf{i}_{\text{ext}}(u) = \sum_{v \in N(u)} \mathbf{i}(u, v)$, and

$$\mathbf{i}_{\text{ext}} = U^T \mathbf{i} = U^T \cdot W \cdot U \cdot \mathbf{v}.$$

If $\mathbf{i}_{\text{ext}}(u) = 0$, we call it a internal node, otherwise, we call it a boundary node.

Graph Laplacian

The matrix $L \triangleq U^T W U$ is again graph Laplacian.

Consider the spectral decomposition of L:

$$L = \sum_{i>1} \lambda_i \mathbf{v}_i \mathbf{v}_i^T.$$

Using the decomposition, the equation becomes to

$$\sum_{i\geq 1} a_i \mathbf{v}_i = \left(\sum_{i>1} \lambda_i \mathbf{v}_i \mathbf{v}_i^T \right) \left(\sum_{i\geq 1} b_i \mathbf{v}_i \right),$$

where $\mathbf{i}_{\text{ext}} = \sum_{i\geq 1} a_i \mathbf{v}_i$ and $\mathbf{v} = \sum_{i\geq 1} b_i \mathbf{v}_i$.

Therefore, we must have $a_1 = 0$, which means the current entering the network is equal to the current leaving the network!

Define the Moore-Penrose pseudo-inverse of L

$$L^+ = \sum_{i>1} \lambda_i^{-1} \mathbf{v}_i \mathbf{v}_i^T.$$

Given $\mathbf{i}_{\text{ext}}$, we can compute $\mathbf{v}$ as long as we can compute L^+.

We shift $\mathbf{v}$ so that

$$\mathbf{v} = L^+ \mathbf{i}_{\text{ext}}.$$

Effective Resistance

We are now able to formally define effective resistance.

$$R_{\mathrm{eff}}(u, v) \triangleq (\mathbf{e}_u - \mathbf{e}_v)^T L^+ (\mathbf{e}_u - \mathbf{e}_v).$$

To see this, assuming one unit of current enters u and leaves v:

$$\mathbf{v} = L^+ (\mathbf{e}_u - \mathbf{e}_v).$$

On the other hand,

$$\mathbf{v}(u) - \mathbf{v}(v) = (\mathbf{e}_u - \mathbf{e}_v)^T \mathbf{v} = (\mathbf{e}_u - \mathbf{e}_v)^T L^+ (\mathbf{e}_u - \mathbf{e}_v).$$

Note that L^{+} is positive semi-definite, we can define

$$L^{+/2} = \sum_{i>1} \lambda^{-1/2} \mathbf{v}_i.$$

Then we can write

$$\mathbf{v}(u) - \mathbf{v}(v) = (\mathbf{e}_u - \mathbf{e}_v)^T \mathbf{v} = (\mathbf{e}_u - \mathbf{e}_v)^T L^{+} (\mathbf{e}_u - \mathbf{e}_v) = \|L^{+/2}(\mathbf{e}_u - \mathbf{e}_v)\|_2^2.$$

Examples: Series and Parallel graphs.

Approximating Effective Resistance

Directly computing effective resistance requires to compute L^+, which is costly.

We can view $L^{+/2}\mathbf{e}_u$ and $L^{+/2}\mathbf{e}_v$ as two vectors in $\mathbb{R}^n$ and approximate their distance using metric embedding technique.

Recall in Lecture 6, we learnt:

Theorem

For any $0 < \varepsilon < \frac{1}{2}$ and any positive integer m, consider a set of m points $S \subseteq \mathbb{R}^n$. There exists an matrix $A \in \mathbb{R}^{k \times n}$ where $k = O\left(\varepsilon^{-2} \log m\right)$ satisfying

$$\forall \mathbf{x}, \mathbf{y} \in S, \quad (1-\varepsilon)\|\mathbf{x}-\mathbf{y}\| \le \|A\mathbf{x} - A\mathbf{y}\| \le (1+\varepsilon)\|\mathbf{x}-\mathbf{y}\|.$$

In our proof of JLT, each entry of the matrix A is from $\mathcal{N}(0, 1/k)$.

We only need to show how to compute $AL^{+/2}$ efficiently...

Let $L' \triangleq W^{1/2}U$, then $(L')^T L' = L^+$.

Therefore $\|L'(\mathbf{e}_u - \mathbf{e}_v)\|_2^2 = R_{\mathrm{eff}}(u, v)$.

We only need to solve d-linear equations in L to obtain $AL'L$.