Algorithms for Big Data (XIII)

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Review

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Ohm's law is used to define the potential drop between two ends of an edge.

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The signed edge-vertex adjacency matrix $U \in \{0, 1, -1\}^{E \times V}$ is defined as

$$U((u, v), w) = \begin{cases} 1 & \text{if } w = u \\ -1 & \text{if } w = v \\ 0 & \text{otherwise.} \end{cases}$$

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Let $W \in \mathbb{R}^{E \times E}$ be $\mathrm{diag}(w(e_1), \ldots, w(e_{|E|}))$.

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If $\mathbf{i}_{\text{ext}}(u) = 0$, we call it a internal node, otherwise, we call it a boundary node.

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Using the decomposition, the equation becomes to

$$\sum_{i\geq 1} a_i \mathbf{v}_i = \left(\sum_{i>1} \lambda_i \mathbf{v}_i \mathbf{v}_i^T \right) \left(\sum_{i\geq 1} b_i \mathbf{v}_i \right),$$

where $\mathbf{i}_{\text{ext}} = \sum_{i\geq 1} a_i \mathbf{v}_i$ and $\mathbf{v} = \sum_{i\geq 1} b_i \mathbf{v}_i$.

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Given $\mathbf{i}_{\text{ext}}$, we can compute $\mathbf{v}$ as long as we can compute L^+.

We shift $\mathbf{v}$ so that

$$\mathbf{v} = L^+ \mathbf{i}_{\text{ext}}.$$

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On the other hand,

$$\mathbf{v}(u) - \mathbf{v}(v) = (\mathbf{e}_u - \mathbf{e}_v)^T \mathbf{v} = (\mathbf{e}_u - \mathbf{e}_v)^T L^+ (\mathbf{e}_u - \mathbf{e}_v).$$

Note that L^+ is positive semi-definite, we can define

$$L^{+/2} = \sum_{i>1} \lambda^{-1/2} \mathbf{v}_i.$$

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Examples: Series and Parallel graphs.

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We can view $L^{+/2}\mathbf{e}_u$ and $L^{+/2}\mathbf{e}_v$ as two vectors in $\mathbb{R}^n$ and approximate their distance using metric embedding technique.

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Recall in Lecture 6, we learnt:

Theorem

For any $0 < \varepsilon < \frac{1}{2}$ and any positive integer m, consider a set of m points $S \subseteq \mathbb{R}^n$. There exists an matrix $A \in \mathbb{R}^{k\times n}$ where $k = O\left(\varepsilon^{-2}\log m\right)$ satisfying

$$\forall \mathbf{x}, \mathbf{y} \in S, \quad (1-\varepsilon)\|\mathbf{x}-\mathbf{y}\| \le \|A\mathbf{x} - A\mathbf{y}\| \le (1+\varepsilon)\|\mathbf{x}-\mathbf{y}\|.$$

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Therefore $\|L'(\mathbf{e}_u - \mathbf{e}_v)\|_2^2 = R_{\mathrm{eff}}(u, v)$.

We only need to solve d-linear equations in L to obtain $AL'L$.