[CS1961: Lecture 11] Lovász Local Lemma

Instructor: Chihao Zhang;
Scribed by Fanghao Shao, Yuehan Wang, Yuchen He

1 Lovász Local Lemma

For a set of bad events $\mathcal{B} = \{B_1, B_2, \dots, B_m\}$ where each $\mathbf{Pr}[B_i] < 1$, the union bound states that $\mathbf{Pr}[\bigcup_{i=1}^m B_i] \leqslant \sum_{i=1}^m \mathbf{Pr}[B_i]$. The equality holds only when the events are disjoint. As a result, $\mathbf{Pr}\left[\bigcap_{i=1}^m \overline{B}_i\right] > 0$ if $\sum_{i=1}^m \mathbf{Pr}[B_i] < 1$. In this case, these events are quite relevant with each other. However, when they are mutually independent, we have $\mathbf{Pr}\left[\bigcap_{i=1}^m \overline{B}_i\right] = \prod_{i=1}^m \mathbf{Pr}\left[\overline{B}_i\right]$, which is always larger than zero. The union bound is quite loose in this situation. This motivates us to introduce a generalized union bound which takes the dependencies between events into consideration.

We define a dependency graph $G = (V, E)$ with $V = [m]$ to reflect the relevance between the events. This graph is required to satisfy that for any $i \in [m]$, B_i is independent with $\{B_j \mid (i, j) \notin E\}$. [1] Let d be the maximum degree of G. Then small d should indicate low correlation between events. For example, when all the events are disjoint, i.e., the union bound achieves the equality, G is a clique. Conversely, when all the events are mutually independent, E can be an empty set. With this dependency graph, we introduce the following lemma.

[1] We does not require the inverse to hold.

Lemma 1 (Lovász Local Lemma) *Suppose for any* $i \in [m]$, $\mathbf{Pr}[B_i] \leqslant p$ *holds for some constant* $p \in [0, 1)$. *If* $ep(d + 1) < 1$, *then* $\mathbf{Pr}\left[\bigcap_{i=1}^m \overline{B}_i\right] > 0$.

Example 1 (k-SAT) *Consider a* k*-CNF with* Δ*-degree*[2]. *We can use the Lovász local lemma to give a sufficient condition for the satisfiability of* ϕ.

We sample each variable u.a.r. from $\{0, 1\}$. *Define* B_j *as the event that the clause* C_j *is not satisfied. Then* $\mathbf{Pr}[B_j] \leqslant 2^{-k}$. *Construct the dependency graph* G *as* $B_i \sim B_j$ *iff* C_i *and* C_j *share variables*[3] *(*$i \neq j$*). Then* $d \leqslant k\Delta$ *since there are* k *variables in each* C_j *and each variable appears in at most* Δ *clauses. By the Lovász local lemma, when* $e \cdot 2^{-k}(k\Delta + 1) < 1$, *with non zero probability, all the clauses can be satisfied and thus* ϕ *is satisfiable.*

[2] A CNF $\phi = C_1 \wedge C_2 \wedge \cdots \wedge C_m$ is k-CNF if each C_j contains k literals. We say the degree of ϕ is Δ if each variable appears in at most Δ clauses in ϕ.

[3] Here $B_i \sim B_j$ means $(i, j) \in E$.

Example 2 (Ramsey Number) *Recall that the Ramsey number* $R(k, k) > n$ *means there exists an edge coloring of* K_n *such that there is no monochromatic* K_k *in* K_n. *For any* $S \in \binom{[n]}{k}$, *define* B_S *as the event that* $G[S]$ *is monochromatic. By the union bound*

$$\mathbf{Pr}\left[\bigcup_{S \in \binom{[n]}{k}} B_S\right] \leqslant \binom{n}{k} 2^{1-\binom{k}{2}} < 1$$

when $n < \left(\frac{1}{e\sqrt{2}} + o(1)\right) k 2^{\frac{k}{2}}$. *Therefore we have* $R(k, k) > \left(\frac{1}{e\sqrt{2}} + o(1)\right) k 2^{\frac{k}{2}}$.

Construct the dependency graph G *as* $B_S \sim B_T$ *iff* $|S \cap T| \geqslant 2$*. Then* $d \leqslant \binom{k}{2}\binom{n-2}{k-2}$*. By the Lovász local lemma, when* $e \cdot \binom{k}{2}\binom{n-2}{k-2}2^{1-\binom{k}{2}} < 1$*, there exists a coloring of* K_n *without monochromatic* K_k*. This yields an improved lower bound of* $R(k,k) > \left(\frac{\sqrt{2}}{e} + o(1)\right) k2^{\frac{k}{2}}$*.*

Example 3 (Independent Set) *Given a graph* $G = (V, E)$*, partition* V *into* $V_1, V_2, \ldots, V_m$ *with each* $|V_i| \geqslant k$ *where* $k = 2e\Delta$ *and* Δ *is the max degree of* G*. We claim that there exists* $u_1, u_2, \ldots, u_m$ *where* $u_i \in V_i$ *such that* $\{u_i\}_{i=1}^m$ *forms an independent set.*

W.l.o.g, we regard each $|V_i|$ *as* k *since we can delete some vertices in* $|V_i| > k$ *and this only strengthens the proposition. Pick* $u_i \in V_i$ *u.a.r. for every* $i \in [m]$*. Define the bad event* B_e *for* $e = (v, w) \in E$ *as both* v *and* w *are picked. Then* $\mathbf{Pr}[B_e] \leqslant \frac{1}{k^2}$*. We construct the dependency graph by letting* $B_{e'} \sim B_e$ *for all the edges* $e' = (v', w')$ *that either* v' *and* v *are in the same part or* w' *and* w *are in the same part. Then* $d < 2k\Delta - 1$*. By the Lovász local lemma, since* $\frac{e}{k^2} \cdot 2k\Delta < 1$*, the probability of the chosen set being an independent set is non-zero.*

2 *Asymmetric Lovász Local Lemma*

Note that in more general cases, $\mathbf{Pr}[B_i]$ can be quite different. Intuitively it is not optimal to use a universal upper bound for $\mathbf{Pr}[B_i]$ in Lemma 1. Therefore, we introduce the asymmetric Lovász local lemma.

Lemma 2 (Asymmetric Lovász Local Lemma) *If there exists* $x \colon \mathcal{B} \to (0,1)$ *that* $\mathbf{Pr}[B_i] \leqslant x(B_i)\prod_{B_j \in N(B_i)}(1 - x(B_j))$*,*[4] *then* $\mathbf{Pr}\left[\cap_{i=1}^m \overline{B}_i\right] \geqslant \prod_{i=1}^m (1 - x(B_i)) > 0$*.*

[4] Here $N(B_i)$ denotes the set of neighbors of B_i in the dependency graph.

Before proving Lemma 2, we show that Lemma 1 can be deduced from this asymmetric Lovász local lemma. Let $x(B_i) = \frac{1}{d+1}$. Then

$$x(B_i)\prod_{B_j \in N(B_i)}(1 - x(B_j)) = \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^{\deg(B_i)} \geqslant \frac{1}{d+1}\left(1 - \frac{1}{d+1}\right)^{d} \geqslant \frac{e^{-1}}{d+1}.$$

Therefore, applying Lemma 2, $pe(d+1) < 1$ indicates that $\mathbf{Pr}\left[\bigcap_{i=1}^m \overline{B}_i\right] > 0$.

Proof of Lemma 2 For $S \subseteq [m]$, let $F_S \triangleq \bigcap_{i \in S} \overline{B}_i$. We claim for all $i \notin S$, $\mathbf{Pr}[B_i \mid F_S] \leqslant x(B_i)$. With this claim, we have that

$$\mathbf{Pr}\left[\bigcap_{i=1}^m \overline{B}_i\right] = \prod_{i=1}^m \mathbf{Pr}\left[\overline{B}_i \,\middle|\, \bigcap_{j<i} \overline{B}_j\right] = \prod_{i=1}^m \left(1 - \mathbf{Pr}\left[B_i \mid F_{[i-1]}\right]\right) \geqslant \prod_{i=1}^m (1 - x(B_i)) > 0$$

and thus the lemma holds. We prove the claim by induction.

In the base case $S = \varnothing$, the claim holds trivially. When S is not empty, divide S into two disjoint sets S_1 and S_2 according to whether the event is adjacent with B_i. Let S_1 contains all the neighbors of B_i in S. If $S_1 = \varnothing$, then all the events in S are independent with B_i. Therefore,

$$\mathbf{Pr}\left[B_i \mid F_S\right] = \mathbf{Pr}\left[B_i\right] \leqslant x(B_i) \prod_{B_j \in N(B_i)} (1 - x(B_j)) \leqslant x(B_i).$$

If S is not empty, w.l.o.g, let $S_1 = \{1, 2, \ldots, r\}$. Then

$$\begin{aligned}\mathbf{Pr}\left[B_i \mid F_S\right] &= \mathbf{Pr}\left[B_i \mid F_{S_1} \wedge F_{S_2}\right] \\ &= \frac{\mathbf{Pr}\left[B_i \wedge F_{S_1} \wedge F_{S_2}\right]}{\mathbf{Pr}\left[F_{S_1} \wedge F_{S_2}\right]} = \frac{\mathbf{Pr}\left[B_i \wedge F_{S_1} \mid F_{S_2}\right]}{\mathbf{Pr}\left[F_{S_1} \mid F_{S_2}\right]}.\end{aligned}$$

Since B_i is independent with F_{S_2}

$$\mathbf{Pr}\left[B_i \wedge F_{S_1} \mid F_{S_2}\right] \leqslant \mathbf{Pr}\left[B_i \mid F_{S_2}\right] = \mathbf{Pr}\left[B_i\right] \leqslant x(B_i) \prod_{B_j \in N(B_i)} (1 - x(B_j)).$$

By induction, we have

$$\begin{aligned}\mathbf{Pr}\left[F_{S_1} \mid F_{S_2}\right] &= \prod_{i=1}^{r} \mathbf{Pr}\left[\overline{B}_i \,\middle|\, \bigcap_{k<i} \overline{B}_k \cap F_{S_2}\right] \\ &= \prod_{i=1}^{r} \left(1 - \mathbf{Pr}\left[B_i \,\middle|\, \bigcap_{k<i} \overline{B}_k \cap F_{S_2}\right]\right) \\ &\geqslant \prod_{i=1}^{r} \left(1 - x\left(B_i\right)\right) \\ &\geqslant \prod_{B_j \in N(B_i)} (1 - x(B_j)).\end{aligned}$$

Therefore,

$$\mathbf{Pr}\left[B_i \mid F_S\right] = \frac{\mathbf{Pr}\left[B_i \wedge F_{S_1} \mid F_{S_2}\right]}{\mathbf{Pr}\left[F_{S_1} \mid F_{S_2}\right]} \leqslant \frac{x(B_i) \prod_{B_j \in N(B_i)} (1 - x(B_j))}{\prod_{B_j \in N(B_i)} (1 - x(B_j))} \leqslant x(B_i).$$

□

Example 4 (Asymmetric Ramsey Number) *Recall that the asymmetric Ramsey number* $R(k, \ell) > n$ *means that there exists a coloring of* K_n *with neither blue* K_k *nor red* K_ℓ. *Consider the situation when* $\ell = 3$, *i.e.,* $R(k, 3)$.

For any $S \in \binom{[n]}{3}$, *define the event* A_S *as* $G[S]$ *is a red* K_3. *For any* $T \in \binom{[n]}{k}$, *define* B_T *as* $G[T]$ *is a blue* K_k. *Correspondingly construct the dependency graph* H. *Let* $d_B(A_S)$ *be the maximum degree of the vertices* $\left\{A_S \,\middle|\, S \in \binom{[n]}{3}\right\}$ *where we only consider their neighbors in* $\left\{B_T \,\middle|\, T \in \binom{[n]}{k}\right\}$. *Similarly define* $d_A(A_S)$, $d_A(B_T)$ *and* $d_B(B_T)$. *Then we have* $d_A(A_S) \leqslant 3(n-3)$ *and* $d_B(A_S) \leqslant \binom{n}{k}$ *and* $d_A(B_T) \leqslant \binom{k}{2}(n-2) < \frac{k^2 n}{2}$ *and* $d_B(B_T) \leqslant \binom{n}{k}$. *We color each edge to red w.p.* p *and to blue w.p.* $1-p$. *Define two mappings* X *and* Y *as* $X(A_S) \equiv x$ *and* $Y(B_T) \equiv y$ *for any* $S \in \binom{[n]}{3}$ *and* $T \in \binom{[n]}{k}$ *where* x *and* y *are constants in* $(0, 1)$.

We want p *to satisfy that for all* A_S, $\mathbf{Pr}[A_S] = p^3 \leqslant x(1-x)^{3(n-3)}(1-y)^{\binom{n}{k}}$ *and for all* B_T, $\mathbf{Pr}[B_T] = (1-p)^{\binom{k}{2}} \leqslant y(1-x)^{\frac{k^2 n}{2}}(1-y)^{\binom{n}{k}}$*. Then by the asymmetric Lovász local lemma, there exists an edge coloring that avoids all the monochromatic events. When* $n = O\left(\frac{k^2}{\log k}\right)$*, the above two formula has feasible solutions* (p, x, y)*. This yields that* $R(k, 3) > c \cdot \frac{k^2}{\log k}$ *for some constant* c*.*

A similar argument gives that $R(k, 4) > k^{5/2+o(1)}$. This is the best ever bound we have known.