[CS1961: Lecture 14] Spectral Graph Theory, Laplacian Matrix, Sensitivity Conjecture

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1 The Spectrum of Adjacent Matrix (cont.)

1.1 Adjacent Matrix and Spectrum

Given an undirected graph $G = (V, E)$, let $A \in \{0,1\}^{n\times n}$ be the adjacent matrix of G. Let $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$ be the n eigenvalues of A with eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. The Courant-Ficher theorem (also called min-max theorem) tells us that

$$\mu_k = \max_{\substack{V\subseteq\mathbb{R}^n \\ \dim(V)=k}} \min_{\mathbf{x}\in V} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} = \max_{\mathbf{x}\perp \mathrm{span}(\mathbf{v}_1,\ldots,\mathbf{v}_{k-1})} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}.$$

Moreover, the eligible $\mathbf{x}$ that maximizes $\frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}}$ is exactly the eigenvector corresponding to μ_k.

In the last lecture, we have shown the relation between the spectrum and some properties of the graph. For example, when G is d-regular, we have $\mu_1 = d$. For a general graph G, μ_1 also reflects the maximum degree of G in a sense.

Theorem 1 $d_{\mathrm{ave}} \leqslant \mu_1 \leqslant d_{\max}$. [1]

[1] Here d_{ave} denotes the average degree and d_{max} denotes the maximum degree.

Proof. The lower bound follows by noticing that

$$\mu_1 = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T A\mathbf{x}}{\mathbf{x}^T\mathbf{x}} \geqslant \frac{\mathbf{1}^T A\mathbf{1}}{\mathbf{1}^T\mathbf{1}} = \frac{\sum_{i,j\in[n]} a_{ij}}{n} = \frac{\sum_i \deg(i)}{n} = d_{\mathrm{ave}}.$$

For the other side, we only need to prove that μ_1 is no larger than the maximum absolute row sum. Let $\mathbf{v}$ be an eigenvector corresponding to μ_1. W.l.o.g., assume $|\mathbf{v}(1)| \geqslant |\mathbf{v}(j)|$ for all $j \in [n]$. Then

$$\mu_1|\mathbf{v}(1)| = |\mu_1\mathbf{v}(1)| = |A\mathbf{v}(1)| = \left|\sum_{i=1}^n a_{1i}\mathbf{v}(i)\right| \leqslant |\mathbf{v}(1)| \sum_{i=1}^n |a_{1i}|.$$

This yields that $\mu_1 \leqslant d_{\max}$. □

1.2 Cauchy Interlacing Theorem

Recall that we use the Cauchy interlacing theorem to give an upper bound for the independent number $\alpha(G)$ in the last lecture. Let $\chi(G)$ be the chromatic number of G. We can also derive an upper bound for $\chi(G)$ by the Cauchy interlacing theorem.

Theorem 2 (Wilf Theorem) $\chi(G) \leqslant \lfloor \mu_1 \rfloor + 1$.

Proof. We prove this by induction on n. When $n = 1$, the bound obviously holds. When $n \geqslant 2$, choose a vertex v with the smallest degree. Therefore,

$$\deg(v) \leqslant d_{\text{ave}} \leqslant \mu_1(G).$$

When the graph G is d-regular, the Wilf theorem indicates that $\chi(G) \leqslant d + 1$, which is tight in this case.

Let H be the graph induced by $V \setminus \{v\}$. By induction hypothesis, we have $\chi(H) \leqslant \lfloor \mu_1(H) \rfloor + 1$. By the Cauchy interlacing theorem, we can further yield that

$$\chi(H) \leqslant \lfloor \mu_1(H) \rfloor + 1 \leqslant \lfloor \mu_1(G) \rfloor + 1.$$

Note that the number of neighbors of v is no larger than $\mu_1(G)$. Therefore, $\lfloor \mu_1(G) \rfloor + 1$ colors are enough to construct a proper coloring. □

2 *Laplacian Matrix*

2.1 *The Spectrum of Laplacian Matrix*

Let $A_G = (w_{ij})_{i,j \in [n]}$ be the adjacent matrix of some graph G (probably weighted) and define $w_i = \sum_{j=1}^{n} w_{ij}$ for all $i \in [n]$. Let $D_G = \text{diag}(w_1, w_2, \ldots, w_n)$. The *Laplacian matrix* of G is defined as $L_G = D_G - A_G$.

With the definition of Laplacian matrix, we can turn to consider the spectrum of L_G instead of A_G. For example, when G is d-regular, $L_G = d\mathbb{I} - A_G$. Let $\lambda_1 \leqslant \lambda_2 \leqslant \cdots \leqslant \lambda_n$ be the eigenvalues of L_G and $\mu_1 \geqslant \mu_2 \geqslant \cdots \geqslant \mu_n$ be the eigenvalues of A_G. Then $\lambda_i = d - \mu_i$ by definition and we have $\lambda_1 = 0$. We claim that this also applies to general graphs.

Lemma 3 $\mathbf{x}^{\mathsf{T}} L_G \mathbf{x} = \sum_{\{i,j\} \in E} w_{ij} (x_i - x_j)^2$.

For a weighted graph G, $E = \{\{i, j\} \mid w_{ij} \neq 0\}$.

Proof. This can be proved by a direct calculation:

$$\begin{aligned}
\sum_{\{i,j\} \in E} w_{ij} (x_i - x_j)^2 &= \sum_{\{i,j\} \in E} w_{ij} \left(x_i^2 - 2x_i x_j + x_j^2\right) \\
&= \sum_{\{i,j\} \in E} w_{ij} \left(x_i^2 + x_j^2\right) - 2 \sum_{\{i,j\} \in E} w_{ij} x_i x_j \\
&= \sum_{i \in V} x_i^2 \sum_{j \sim i} w_{ij} + \sum_{i \in V} x_i^2 w_{ii} - 2 \sum_{\{i,j\} \in E} w_{ij} x_i x_j \\
&= \sum_{i \in V} x_i^2 w_i + \sum_{i \in V} x_i^2 w_{ii} - \left(\sum_{i,j \in V} w_{ij} x_i x_j + \sum_{i \in V} w_{ii} x_i^2 \right) \\
&= \sum_{i \in V} x_i^2 w_i - \sum_{i,j \in V} w_{ij} x_i x_j \\
&= \mathbf{x}^{\mathsf{T}} D_G \mathbf{x} - \mathbf{x}^{\mathsf{T}} A_G \mathbf{x} = \mathbf{x}^{\mathsf{T}} L_G \mathbf{x}.
\end{aligned}$$

□

Equipped with Lemma 3, we then prove our claim.

Claim 4 *For any graph* G *with* $w_{ij} \geqslant 0$*,* $\lambda_1(L_G) = 0$*.*

Proof. By the min-max theorem,

$$\lambda_1(L_G) = \min_{x \neq 0} \frac{\mathbf{x}^T L_G \mathbf{x}}{\mathbf{x}^T \mathbf{x}} = \min_{x \neq 0} \frac{\sum_{\{i,j\} \in E} w_{ij} (x_i - x_j)^2}{\sum_{i=1}^n x_i^2} \geqslant 0$$

where the second equation follows from Lemma 3. Furthermore,

$$\lambda_1(L_G) \leqslant \frac{\mathbf{1}^T L_G \mathbf{1}}{\mathbf{1}^T \mathbf{1}} = 0.$$

Therefore, we have $\lambda_1(L_G) = 0$. □

Example 1 (Complete Graph) *When* G *is a complete graph* K_n*,*

$$L_G = \begin{bmatrix} n-1 & 0 & \cdots & & \\ 0 & n-1 & & & \\ \vdots & & & \ddots & 0 \\ & & & 0 & n-1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & \cdots & & \\ 1 & 0 & & & \\ \vdots & & & \ddots & 1 \\ & & & 1 & 0 \end{bmatrix}.$$

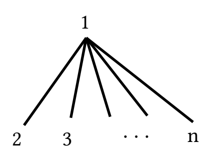
Pick $\mathbf{v} \perp \mathbf{1}$*. That is,* $\sum_{i=1}^n \mathbf{v}(i) = 0$*, or equivently,* $v(1) = -\sum_{i=2}^n \mathbf{v}(i)$*. Then*

$$L_G \mathbf{v}(1) = (n-1)\mathbf{v}(1) - \sum_{i=2}^n \mathbf{v}(i) = n\mathbf{v}(1).$$

Similarly we have $L_G \mathbf{v}(i) = n\mathbf{v}(i)$ *for every other* $i \in [n]$ *and thus* $L_G \mathbf{v} = n\mathbf{v}$ *for all* $\mathbf{v} \perp \mathbf{1}$*. Therefore, the spectrum of* L_{K_n} *is* $0, n, n, \ldots, n$*, which respectively corresponds to the eigenvectors* $\mathbf{1}$ *and the* $n-1$ *independent vectors that are perpendicular to* $\mathbf{1}$*.*

Example 2 (Star Graph) *When* G *is a star,*

$$L_G = \begin{bmatrix} n-1 & 0 & \cdots & \\ 0 & 1 & & \\ \vdots & & \ddots & 0 \\ & & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & & \\ \vdots & & \ddots & 0 \\ 1 & & 0 & 0 \end{bmatrix} = \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & 1 & & \\ \vdots & & \ddots & 0 \\ -1 & & 0 & 1 \end{bmatrix}.$$



Let $\mathbf{e}_i = (0, 0, \ldots, 0, 1, 0, \ldots, 0)$ *be a unit vector where only the* i*-th entry is* 1*. Then for every* $i, j \geqslant 2$ *and* $i \neq j$*,* $e_i - e_j$ *is an eigenvector of* L_G *with eigenvalue* 1*. Since* $\dim\left(\text{span}\left(\{e_i - e_j\}_{\substack{i \neq j \\ i,j \geqslant 2}}\right)\right) = n-2$*, it only needs to determine the remaining one eigenvalue (we have already known that* $\lambda_1 = 0$*). Note that*

$$\text{Trace}(L_G) = n - 1 + n - 1 = \sum_{i=1}^n \lambda_i = 0 + (n-2) + \lambda_n.$$

Therefore, we have $\lambda_n = n$*. It is easy to verify that the eigenvector corresponds to* λ_n *is* $[1-n \quad 1 \quad \cdots \quad 1]^T$*.*

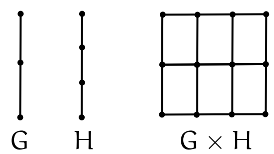
2.2 Product of Graphs

The spectrum of Laplacian matrix is useful when analysing the properties of the graph. However, sometimes it might be difficult to calculate. The technique of graph product can help to calculate the spectrum of some specific kinds of graphs.

Given two graphs $G = (V_G, E_G)$ and $H = (V_H, E_H)$, we can define the product $G \times H = (V_G \times V_H, E_{G\times H})$ as

$$E_{G\times H} = \{\{(a,b),(a,b')\} \mid \{b,b'\} \in E_H\} \cup \{\{(a',b),(a,b)\} \mid \{a,a'\} \in E_G\}.$$

For example, the product of two chains forms a grid.



Let $\lambda_1, \lambda_2, \ldots, \lambda_n$ be the eigenvalues of L_G corresponding to eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and $\mu_1, \mu_2, \ldots, \mu_m$ be the eigenvalues of L_H corresponding to eigenvectors $\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_m$. Then $r_{ij} = \lambda_i + \mu_j$ is an eigenvalue of $L_{G\times H}$ with eigenvector $\mathbf{w}_{ij} = \mathbf{v}_i \otimes \mathbf{u}_j$.[2]

[2] Let $\mathbf{v} = [v_1, v_2, \ldots, v_n]^T$. Then

$$\mathbf{v} \otimes \mathbf{u} = \begin{bmatrix} v_1 \cdot \mathbf{u} \\ v_2 \cdot \mathbf{u} \\ \vdots \\ v_n \cdot \mathbf{u} \end{bmatrix}.$$

2.3 Hoffman's Bound

By analysing the spectrum of the Laplacian matrix, we can deduce some properties in the graph.

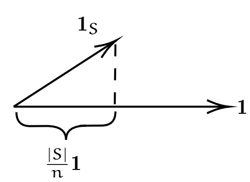
Theorem 5 (Hoffman's Bound) *For any independent set* S *of* G, $|S| \leqslant n\left(1 - \frac{d_{ave}(S)}{\lambda_n}\right)$.[3]

[3] Here $d_{ave}(S)$ is the average degree of the vertices in S.

Proof. Let $\mathbf{1}_S$ be the indicator vector of S. By Lemma 3,

$$\mathbf{1}_S^T L_G \mathbf{1}_S = \sum_{\{i,j\}\in E} (\mathbf{1}_S(i) - \mathbf{1}_S(j))^2 = |S| \cdot d_{ave}(S).$$

Since $\mathbf{x}^T L_G \mathbf{x} = \sum_{\{i,j\}\in E} (\mathbf{x}(i) - \mathbf{x}(j))^2$, the value of $\mathbf{x}^T L_G \mathbf{x}$ does not change if we add a constant offset to $\mathbf{x}$. Pick $\mathbf{x} = \mathbf{1}_S - c\mathbf{1}$ where c is the minimizer of $\|\mathbf{1}_S - c\mathbf{1}\|^2$. As the figure shows, the best choice for c is $c = \frac{|S|}{n}$.



Then

$$\lambda_n = \max_{\mathbf{x}\neq 0} \frac{\mathbf{x}^T L_G \mathbf{x}}{\mathbf{x}^T \mathbf{x}} \geqslant \frac{\mathbf{1}_S^T L_G \mathbf{1}_S}{\langle \mathbf{1}_S - c\mathbf{1}, \mathbf{1}_S - c\mathbf{1}\rangle} = \frac{|S| \cdot d_{ave}(S)}{\|\mathbf{1}_S - \frac{|S|}{n}\mathbf{1}\|^2}.$$

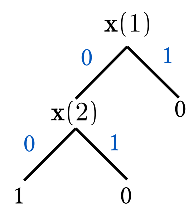
Direct calculation yields that $|S| \leqslant n\left(1 - \frac{d_{ave}(S)}{\lambda_n}\right)$. □

The Hoffman's bound on d-regular graphs indicates that $\alpha(G) \leqslant n\left(1 - \frac{d}{\lambda_n}\right)$. When G is a d-regular bipartite graph, we have $\lambda_n(L_G) = 2d$. Therefore, we have $\alpha(G) \leqslant \frac{n}{2}$ by the Hoffman's bound. The bound is tight in this case.

3 *Sensitivity Conjecture*

3.1 *Boolean Function and Its Sensitivity*

Let $f\colon \{-1,1\}^n \to \{0,1\}$ be a boolean function. We can use a decision tree to describe f and the depth of the tree measures the complexity of f. For example, consider the function f that $f(\mathbf{x}) = 1$ iff $\mathbf{x}(1) = \mathbf{x}(2) = 0$. The decision tree for this function is



In contrast, when $f(\mathbf{x}) = \begin{cases} 1, & \text{if } \left(\sum_i \mathbf{x}(i)\right) \mod 2 = 1 \\ 0, & o.w. \end{cases}$, the depth of the decision tree is n, which means the function is much more complex.

There is another way to measure the complexity of f. We can find a polynomial p such that $p(\mathbf{x}) = f(\mathbf{x})$ for any $\mathbf{x} \in \{-1,1\}^n$. Larger degree of p indicates that f is more complex.

For every $\mathbf{x} \in \{-1,1\}^n$, let

$$s(f,\mathbf{x}) = |\{i \mid f(\mathbf{x}) \neq f(\mathbf{x} \oplus i)\}|$$

where $\mathbf{x} \oplus i$ means flipping the i-th bit of $\mathbf{x}$. The sensitivity of f is defined as $s(f) \triangleq \max_{\mathbf{x}\in\{-1,1\}^n} s(f,\mathbf{x})$. Just as indicated by its name, this quantity reflects the sensitivity to perturbations of f.

Similarly, we can define block sensitivity. Define $bs(f,\mathbf{x})$ as the maximum number t of disjoint subsets $B_1,\ldots,B_t \subseteq [n]$ such that $f(\mathbf{x}) \neq f(x \oplus B_i)$ for every $i \in [t]$. The block sensitivity of f is $bs(f) \triangleq \max_{\mathbf{x}\in\{-1,1\}^n} bs(f,\mathbf{x})$. Obviously, $s(f) \leqslant bs(f)$.

3.2 *Sensitivity Conjecture*

The sensitivity conjecture states that there exists positive constant c such that $bs(f) \leqslant (s(f))^c$. In other words, $bs(f)$ and $s(f)$ are equivalent in a polynomial sense.

Let Q_n be the n-dimensional hypercube[4]. In the work of [GL92] this conjecture has been reduced to the following proposition.

[4] A hypercube $Q_n = (V,E)$ is a graph with $V = \{0,1\}^n$ and $E = \{\{x,y\} \mid x = y \oplus i \text{ for some } i\}$

Proposition 6 *For any induced subgraph* H *of* Q_n *with* $2^{n-1}+1$ *vertices,* $\Delta(H) \geqslant \sqrt{n}$.[5]

[5] $\Delta(H)$ is the maximum degree of H.

Huang Hao [Hua19] gave a remarkable proof of the above proposition and hence the sensitivity conjecture.

The $2^{n-1}+1$ vertices can not be reduced any more in Proposition 6. Note that hypercube is bipartite. So there exists an independent set with 2^{n-1} vertices in Q_n.

Proof. Let $A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}$ for $n \geqslant 2$. We claim that $A_n^2 = nI$. This can be proved by induction. When $n = 1$, it is trivial to have $A_1^2 = I$. For $n \geqslant 2$,

$$A_n^2 = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}^T \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix} = \begin{bmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{bmatrix} = nI.$$

This indicates that the eigenvalue of A_n is either $\sqrt{n}$ or $-\sqrt{n}$. Note that the trace of A_n is 0. Therefore,

$$\lambda_1 = \cdots = \lambda_{2^{n-1}} = \sqrt{n} \quad \text{and} \quad \lambda_{2^{n-1}+1} = \cdots = \lambda_{2^n} = -\sqrt{n}.$$

It can be verified that A_n is a signed adjacent matrix of Q_n. That is, $A_n(x, y) \neq 0$ indicates that $Q_n(x, y) \neq 0$ and $A_n(x, y) \in \{0, \pm 1\}$. Pick the columns and rows in H, we can get a principal submatrix of A_n, denoted as $A_n(H)$. Then we have $\Delta(H) \geqslant \lambda_1(A_n(H))$. By the Cauchy interlacing theorem, $\lambda_1(A_n(H)) \geqslant \lambda_{2^{n-1}}(A_n) = \sqrt{n}$. This completes the proof. □

References

[GL92] Craig Gotsman and Nathan Linial. The equivalence of two problems on the cube. *Journal of Combinatorial Theory, Series A*, 61(1):142–146, 1992. 5

[Hua19] Hao Huang. Induced subgraphs of hypercubes and a proof of the sensitivity conjecture. *Annals of Mathematics*, 190(3):949–955, 2019. 5