[CS1961: Lecture 2] Counting Techniques, Combinatorial Proof, Inclusion-Exclusion Principle

Instructor: Chihao Zhang;
Scribed by Shuze Chen, Haosong Liu, Yuehan Wang, Yuchen He

1 Counting Techniques

1.1 Balls-into-Bins

Recall the password counting problem we discussed in the last lecture. Different restrictions will lead to distinct results. In this lecture, we will introduce the model of balls-into-bins, which is more general and can encode more natural counting problems. Suppose we are throwing k balls into n bins. Consider the following various settings.

① k **distinct balls into** n **distinct bins**: This is equivalent with counting the number of length-k passwords with size-n alphabets. For each ball, there are n distinct choices. So the number of possible ways is n^k.

② k **distinct balls into** n **distinct bins, no bin contains more than one ball**: This is equivalent with counting the number of length-k passwords with size-n alphabets without repetition. After choosing bins for the first j balls, the $(j+1)$-th ball only has $n-j$ choices. In this situation, there are $(n)_k$ possible ways to distribute the balls.

$(n)_k \triangleq n \cdot (n-1) \cdot (n-2) \dots (n-k+1)$.

③ k **identical balls into** n **distinct bins, no bin contains more than one ball**: This is the same with setting a length-k password with size-n alphabets while not allowing repetition and not caring about the order. This problem can be regarded as picking k bins out of n bins, which has $\binom{n}{k}$ possible ways in total.

$\binom{n}{k} \triangleq \frac{n!}{(n-k)!k!}$.

④ k **identical balls into** n **distinct bins**: This is equivalent with setting a length-k password with size-n alphabets while not caring about the order. That is, we need to choose a multi-set of size k from $[n]$. There are $\left(\!\binom{n}{k}\!\right)$ choices in total.

$\left(\!\binom{n}{k}\!\right) \triangleq \binom{n+k-1}{k}$.

1.2 Rota's Twelvefold Way

Actually, we can list twelve different settings based on whether the balls and bins are distinct and the restrictions of the number of balls in each bin as shown in Table 1. This is called Rota's twelvefold ways.

The first four situations have been analysed in Section 1.1.

⑤ k **identical balls into** n **distinct bins, no empty bins**: We can first put one ball in each bin and reduce the problem to throwing $k-n$ identical balls into n distinct bins with no restriction.

		How many balls allowed in each bin?		
k balls	n bins	no restriction	≤ 1	≥ 1
distinct	distinct	① n^k	② $(n)_k$	⑦ $n!S(k,n)$
identical	distinct	④ $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$	③ $\binom{n}{k}$	⑤ $\left(\binom{n}{k-n}\right) = \binom{k-1}{n-1}$
distinct	identical	⑧ $\sum_{i=1}^{n} S(k,i)$	1 or 0	⑥ $S(k,n)$
identical	identical	⑩ $\sum_{i=1}^{n} P(k,i)$	1 or 0	⑨ $P(k,n)$

Table 1: Rota's Twelvefold Way

Or we can imagine the problem as inserting n boards into a row of k balls where the balls between the i-th and $(i+1)$-th board represents the balls in the $(i+1)$-th bin. Then it is equivalent with counting the binary strings with n 1's and k 0's that ends up with 1, the total number of which is $\left(\binom{n}{k-n}\right)$.

⑥ k **distinct balls into** n **identical bins, no empty bins**: This is equivalent to partitioning a k-size set into exactly n disjoint non-empty groups, which has $S(k,n)$ ways in total[1].

[1] $S(k,n)$ is Stirling number of second kind.

⑦ k **distinct balls into** n **distinct bins, no empty bins**: This is to count the number of onto mappings $f\colon [k] \to [n]$. We can first partition the k balls into n disjoint non-empty groups and then number the groups from 1 to n. So there are $n!S(k,n)$ ways in total.

⑧ k **distinct balls into** n **identical bins**: This is to find the number of ways to partition a k-size set into at most n disjoint non-empty parts. We can use the result in ⑥ and get the answer $\sum_{i=1}^{n} S(k,i)$ by enumerating the number of non-empty bins.

⑨ k **identical balls into** n **identical bins, no empty bins**: This is to partition k into n positive numbers, which has $P(k,n)$ ways in total[2].

[2] $P(k,n)$ is called the integer partition number which is defined as the number of ways to partition k into n positive numbers. It has no closed form but can be computed by recursion.

⑩ k **identical balls into** n **identical bins**: This is equivalent with partitioning k into at most n positive numbers. By enumerating the number of non-empty bins, we know from ⑨ that there are $\sum_{i=1}^{n} P(k,i)$ ways in total.

2 *Combinatorial Proof*

Combinatorial counting reflects certain numerical relationships. This inspires us to use the combinatorial interpretation rather than algebraic methods to prove formulas. Most identities are established via *double counting*.

Proposition 1 $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$.

Proof. To throw k identical balls into n distinct bins with no bin contains more than one ball, we can first decide whether to choose the first bin. If we do so, then the problem is reduced to throwing $k-1$ identical balls into $n-1$ distinct bins with no bin contains more than one ball. If the first bin is not chosen, then all the k balls will be thrown into the remaining $n-1$ bins. Thus we have $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$. □

Proposition 2 $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n}{k-1}\!\right) + \left(\!\binom{n-1}{k}\!\right)$.

Proof. To choose a multi-set of size k from $[n]$, we can first decide whether 1 is chosen to be in the set. If so, then it remains to choose a multi-set of size $k-1$ from $[n]$. If 1 is not in the set, then the k items are all chosen from $[n] \setminus \{1\}$. Thus we have $\left(\!\binom{n}{k}\!\right) = \left(\!\binom{n}{k-1}\!\right) + \left(\!\binom{n-1}{k}\!\right)$. □

Proposition 3 (Vandermonde's Formula)

$$\forall m, n, k > 0, \quad \sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k}$$

Proof. Suppose there are m boys and n girls in the class. To choose k students, we can choose j boys and $k-j$ girls respectively. By enumerating all possible j's, we have $\sum_{j=0}^{k} \binom{m}{j}\binom{n}{k-j} = \binom{m+n}{k}$. □

Proposition 4 $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$.

Proof. When partitioning $[n]$ into k parts, we can first decide whether n monopolizes a part. If so, then the the remaining $[n-1]$ will be assigned to the other $k-1$ parts which has $S(n-1,k-1)$ possible ways. If not, we first partition $[n-1]$ into k non-empty parts and then assign n to an arbitrary part. Thus, we have $S(n,k) = S(n-1,k-1) + k \cdot S(n-1,k)$. □

Proposition 5

$$S(n,k) = \sum_{j=0}^{n-1} \binom{n-1}{j} \cdot S(n-j-1, k-1).$$

Proof. When partitioning $[n]$ into k parts, we can enumerate the elements that are in the same part with n. If there are $j+1$ elements in the part that contains n, then the other $n-j-1$ elements will be partitioned into the remaining $k-1$ parts. Thus, $S(n,k) = \sum_{j=0}^{n-1} \binom{n-1}{j} \cdot S(n-j-1,k-1)$. □

Proposition 6 $P(n,k) = P(n-1,k-1) + P(n-k,k)$

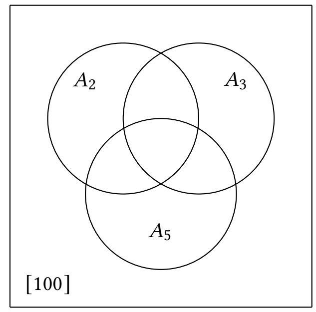
Proof. Partitioning n into k positive numbers is to represent n as $n = a_1 + a_2 + \cdots + a_k$ where $1 \le a_1 \le a_2 \le \cdots \le a_k$. Note that if $a_1 = 1$, the problem is reduced to partition $n-1$ into $k-1$ positive numbers. If not, then each a_i is no less than 2. We can subtract 1 from each addend and then partition the remaining $n-k$ into k positive integers, which has $P(n-k,k)$ possible ways. Thus, $P(n,k) = P(n-1,k-1) + P(n-k,k)$. □

Proposition 7

$$P(n,k) = \sum_{j=1}^{k} P(n-k, j)$$

Proof. Assume that $1 = a_1 = a_2 = \cdots = a_i < a_{i+1}$. Then representing n as $a_1 + a_2 + \cdots + a_k$ is equivalent with representing $n - k$ as $(a_{i+1} - 1) + (a_{i+2} - 1) + \cdots + (a_k - 1)$ where $1 \leq (a_{i+1} - 1) \leq (a_{i+2} - 1) \leq \cdots \leq (a_k - 1)$. The problem is reduced to counting $P(n-k, k-i)$ once i is fixed. Thus we have $P(n,k) = \sum_{j=1}^{k} P(n-k, j)$. □

3 Inclusion-Exclusion Principle



Example 1 *Consider the problem to count the numbers in* $[100]$ *that are not divisible by 2,3,5. Let* A_i *be the set of numbers in* $[100]$ *that is divisible by* i*. Then as the Venn diagram shows, the answer is*

$$\begin{aligned} & 100 - |A_2| - |A_3| - |A_5| + |A_2 \cap A_3| + |A_3 \cap A_5| + |A_2 \cap A_5| - |A_2 \cap A_3 \cap A_5| \\ = \; & 100 - 50 - 33 - 20 + 16 + 6 + 10 - 3 \\ = \; & 26. \end{aligned}$$

Let U be the universe set. Suppose we have a set of bad properties $P = \{A_1, A_2, \cdots, A_m\}$ where $A_i \subseteq U$. For any $J \subseteq P$, $N_=(J) \triangleq \left|\{x \in U | (\forall A \in J, x \in A) \wedge (\forall A \in P \backslash J, x \notin A)\}\right|$ and $N_\geq(J) \triangleq \left|\{x \in U | \forall A \in J, x \in A\}\right| = \left|\bigcap_{A \in J} A\right|$ [3].

[3] For example, when $J = \{A_1, A_2\}$, $N_=(J)$ is the number of items in $A_1 \cap A_2$ while not in $A_3 \cup A_4 \cup \cdots \cup A_m$ and $N_\geq(J)$ is the number of items in $A_1 \cap A_2$.

Theorem 8 (Inclusion-Exclusion Principle)

$$N_=(\phi) = \sum_{J \subseteq P} (-1)^{|J|} N_\geq(J) = \sum_{j=0}^{m} (-1)^j \sum_{J \in \binom{P}{j}} N_\geq(J)$$

The notation $\binom{P}{j}$ means the subsets of P that contains exactly j items.

Proof. Note that $N_=(\phi) = |\bigcap_{i=1}^{m} \overline{A}_i|$ and $N_\geq(J) = |\bigcap_{A_j \in J} A_j|$. Then we only need to prove

$$|\bigcap_{i=1}^{m} \overline{A}_i| = \sum_{J \subseteq P} (-1)^{|J|} |\bigcap_{A_j \in J} A_j|.$$

For any $x \in U$ that satisfies none of A_i, it appears once in the LHS. In the RHS, it is only counted when $J = \emptyset$.

For any $x \in U$ that satisfies some A_i, it is not counted in $|\bigcap_{i=1}^{m} \overline{A}_i|$. Without loss of generality, assume x satisfies and only satisfies $A_1, A_2, \ldots, A_n$. Then the number of its occurrence in the RHS is

$$\sum_{J \subseteq \{A_1, A_2, \ldots, A_n\}} (-1)^{|J|} = \sum_{j=0}^{n} \binom{n}{j} (-1)^j = 0.$$

□

Example 2 (Permutations without fixed points) *Let* A_i *be the set of permutations of* $[n]$ *that for any* $f \in A_i$, $f(i) = i$*, i.e.,* i *is a fixed point of the* *permutations in A_i. Let $P = \{A_1, A_2, \ldots, A_n\}$. Then the number of permutations without fixed points is $N_=(\emptyset)$. By the inclusion-exclusion principle,*

$$\begin{aligned} N_=(\emptyset) &= \sum_{J \subseteq P} (-1)^{|J|} N_\geq(J) \\ &= \sum_{j=0}^{n} \sum_{J \in \binom{P}{j}} (-1)^j N_\geq(J) \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)!\ . \end{aligned}$$

Example 3 (Stirling number of the second kind) *Let f be an onto mapping from $[k]$ to $[n]$. By the analysis in Section 1.2, we know that there are $n!S(k,n)$ such onto mappings in total.*

Let $A_i = \left\{ f \mid f^{-1}(i) = \emptyset \right\}$ for $i \in [n]$. By the inclusion-exclusion principle, the number of onto mappings is

We write $J = \left\{ A_{i_1}, A_{i_2}, \ldots, A_{i_k} \right\}$ as $\{i_1, i_2, \ldots, i_k\}$ for brevity. For example, $N_\geq(\{1,3,5\})$ is the number of mappings from $[k]$ to $[n]$ that there are no preimages for 1, 3 and 5.

$$\begin{aligned} N_=(\emptyset) &= \sum_{J \subseteq [n]} (-1)^{|J|} N_\geq(J) \\ &= \sum_{j=0}^{n} \sum_{J \in \binom{[n]}{j}} (-1)^j N_\geq(J) \\ &= \sum_{j=0}^{n} \binom{n}{j} (-1)^j (n-j)^j\ . \end{aligned}$$

Thus we have

$$S(k,n) = \frac{1}{n!} \sum_{j=0}^{n} (-1)^j \binom{n}{j} (n-j)^j.$$