[CS1961: Lecture 3] Ordinary generating functions

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1 Ordinary generating functions

1.1 The Example of Changing Note

Suppose there are banknotes with denominations of ¥1, ¥5, ¥10 and ¥20. How many ways to are there to change ¥30?

- If we only use ¥1, all the different amounts we can scrape together is

$$A = \varnothing + \textcircled{1} + \textcircled{1}\textcircled{1} + \textcircled{1}\textcircled{1}\textcircled{1} + \cdots .$$

With exponents denoting the number of banknotes, we can rewrite it as

$$A = \varnothing + \textcircled{1} + \textcircled{1}^2 + \textcircled{1}^3 + \cdots .$$

Here we use $\varnothing$ to represent ¥0. Two concatenate $\textcircled{1}$'s means we use two ¥1 to compose ¥2, and so on.

- If we only use ¥1 and ¥5, the amounts we can compose is

$$\begin{aligned} B &= \varnothing + \textcircled{5}A + \textcircled{5}^2A + \textcircled{5}^3A + \cdots \\ &= \left(\varnothing + \textcircled{5} + \textcircled{5}^2 + \textcircled{5}^3 + \cdots\right)A. \end{aligned}$$

- In a similar way, if we use ¥1, ¥5 and ¥10, the amounts we can compose is

$$C = \left(\varnothing + \textcircled{10} + \textcircled{10}^2 + \textcircled{10}^3 + \cdots\right)B.$$

- If we can use all the banknotes: ¥1, ¥5, ¥10 and ¥20, all possible amounts we can compose is

$$\begin{aligned} D &= \left(\varnothing + \textcircled{20} + \textcircled{20}^2 + \textcircled{20}^3 + \cdots\right)C \\ &= \left(\varnothing + \textcircled{20} + \textcircled{20}^2 + \textcircled{20}^3 + \cdots\right)\cdot\left(\varnothing + \textcircled{10} + \textcircled{10}^2 + \textcircled{10}^3 + \cdots\right) \\ &\quad \cdot\left(\varnothing + \textcircled{5} + \textcircled{5}^2 + \textcircled{5}^3 + \cdots\right)\cdot\left(\varnothing + \textcircled{1} + \textcircled{1}^2 + \textcircled{1}^3 + \cdots\right). \end{aligned}$$

Furthermore, we write $\textcircled{1}$, $\textcircled{5}$, $\textcircled{10}$ and $\textcircled{20}$ as z, z^5, z^{10} and z^{20} respectively. Then

$$\begin{aligned} D = &\left(1 + z^{20} + z^{40} + z^{60} + \cdots\right)\cdot\left(1 + z^{10} + z^{20} + z^{30} + \cdots\right) \\ &\cdot\left(1 + z^5 + z^{10} + z^{15} + \cdots\right)\cdot\left(1 + z + z^2 + z^3 + \cdots\right). \end{aligned}$$

It's clear that the coefficient of z^{30} in D, written as $[z^{30}]D$, equals to the number of ways to change ¥30. When $|z| < 1$, D is convergent[1] and equals to

$$\frac{1}{(1-z)(1-z^5)(1-z^{10})(1-z^{20})}.$$

The coefficient $[z^m]D$ can be calculated using Taylor expansion. That is, $[z^m]D(z) = \frac{D(z)^{(m)}}{m!}|_{z=0}$.

[1] Actually, we do not require the generating function to converge. In fact, the generating function is not treated as a real function and the variable z is an abstraction that we do not care about its concrete value.

1.2 Definition and Basic Rules

With the intuition given in Section 1.1, we then introduce the formal definition of the ordinary generating function.

Definition 1 (Ordinary generating functions) *Given a sequence* $\{a_n\}_{n\geq 0}$*, the ordinary generating function with regard to* $\{a_n\}_{n\geq 0}$ *is* $A(z) = \sum_{n\geq 0} a_n z^n$*. The coefficient of the* n*-order term in* A *is written as* $[z^n]A$*, which equals to* a_n*.*

Here are some frequently used basic series.

- $a_k = 1$, $A(z) = 1 + z + z^2 + \cdots = \frac{1}{1-z}$;
- $a_k = \binom{n}{k}$, $A(z) = \binom{n}{0} + \binom{n}{1}z + \binom{n}{2}z^2 + \cdots = \sum_{k\geq 0}\binom{n}{k}z^k = (1+z)^n$;
- $a_k = \left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$, $A(z) = \left(\!\binom{n}{0}\!\right) + \left(\!\binom{n}{1}\!\right)z + \cdots = \sum_{k\geq 0}\left(\!\binom{n}{k}\!\right)z^k = \frac{1}{(1-z)^n}$.

Given two basic generating functions $F(z) = \sum_{k\geq 0} f_k z^k$ and $G(z) = \sum_{k\geq 0} g_k z^k$, we can operate on them based on the following basic rules.

- $[z^k](F+G) = f_k + g_k$;
- $[z^k](z^m F) = f_{k-m}$;
- $[z^k](c \cdot F) = cf_k$;
- $[z^k](F') = (k+1)f_{k+1}$.

2 Applications

2.1 Solving Recurrence

Consider the Fibonacci sequence defined as

- $f_0 = 0, f_1 = 1$;
- $\forall n \geq 2, f_n = f_{n-1} + f_{n-2}$.

The corresponding generating function is

$$F(z) = f_0 + f_1 z + f_2 z^2 + f_3 z^3 + f_4 z^4 + \cdots .$$

Multiply both sides by z and z^2 respectively. We have

$$\begin{aligned} F(z) &= f_0 + f_1 z + f_2 z^2 + f_3 z^3 + f_4 z^4 + \cdots \\ zF(z) &= \qquad f_0 z + f_1 z^2 + f_2 z^3 + f_3 z^4 + \cdots \\ z^2F(z) &= \qquad\qquad f_0 z^2 + f_1 z^3 + f_2 z^4 + \cdots \end{aligned}$$

Note that $F(z) - zF(z) - z^2F(z) = f_0 + f_1 z - f_0 z = z$. Then we have $F(z) = \frac{z}{1-z-z^2}$. The problem to compute f_n is reduced to calculating $[z^n]\left(\frac{z}{1-z-z^2}\right)$.

The idea is to represent $\frac{z}{1-z-z^2}$ as the combination of basic series. Assume that

$$F(z) = \frac{z}{1-z-z^2} = \frac{A}{1-c_1 z} + \frac{B}{1-c_2 z}$$

for some constant c_1, c_2, A and B. Note that the two roots of the equation $1-z-z^2 = 0$ are $z_1 = -\frac{1+\sqrt{5}}{2}$ and $z_2 = -\frac{1-\sqrt{5}}{2}$. Then we can work out that $c_1 = \frac{1+\sqrt{5}}{2}$ and $c_2 = \frac{1-\sqrt{5}}{2}$ since $1-z-z^2 = (1-c_1 z)\cdot(1-c_2 z)$. Sequentially, we can work out that $A = \frac{1}{\sqrt{5}}$ and $B = -\frac{1}{\sqrt{5}}$ from the equations

$$\begin{cases} A+B=0 \\ Ac_2 + Bc_1 = -1 \end{cases}.$$

Thus we have

$$f_n = [z^n]\left(\frac{z}{1-z-z^2}\right) = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

From above example, we can summarize a recipe for solving recurrence:

(1) Write down the equation that holds for any $n \in \mathbb{Z}$.[2] For example, for the Fibonacci Sequence, it holds for all n that $f_n = f_{n-1}+f_{n-2}+\mathbb{1}[n = 1]$.

(2) Multiply both sides by z^n, take sum for all $n \in Z$ and write both sides in terms of $F(z)$. That is,

$$\begin{aligned} & f_n z^n = f_{n-1}z^n + f_{n-2}z^n + \mathbb{1}[n=1]z^n, \\ \Rightarrow & \sum_n f_n z^n = \sum_n f_{n-1}z^n + \sum_n f_{n-2}z^n + z, \\ \Rightarrow & F(z) = zF(z) + z^2F(z) + z. \end{aligned}$$

(3) Solve $F(z)$.

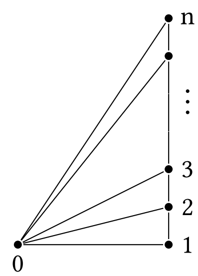
(4) Expand $F(z)$ and read off its coefficient.

[2] W.l.o.g, assume that $f_n = 0$ for $n < 0$.

Example 1 (Spanning trees in a fan) *A fan is a graph with the following structure. Let f_n be the number of spanning trees in a fan with $n+1$ vertices. Then it is easy to know that $f_1 = 1$, $f_2 = 3$ and $f_3 = 8$. For a general n, we can use the ordinary generating function to compute f_n.*

First we deduce the recursion formula of f_n. Note that in a spanning tree of an $(n+1)$-vertices fan:

- *Assume that the vertex n is not connected to the vertex 0 , then it must be connected to vertex $(n-1)$. Based on this assumption, there are f_{n-1} possible spanning trees.*
- *Assume that n is connected to vertex 0. Let k be the smallest index that there are direct edges between $(n, n-1)$, $(n-1, n-2)$, and $(k+1, k)$.*



Once k *is fixed, there are* f_{k-1} *kinds of spanning trees of the remaining fan with* k *vertices. Enumerating all* $k \in [n] \setminus \{1\}$*, there are* $\sum_{k \le n} f_{k-1}$ *ways to construct the spanning trees in this form. In particular, when* $k = 1$*, the chain itself is a spanning tree. So we need to add an additional* 1 *if* $n > 0$*.*

Therefore, we get the equation

$$f_n = f_{n-1} + \sum_{k<n} f_k + \mathbb{1}[n > 0].$$

Multiply both sides by z^n *and take sum for all* $n \in \mathbb{Z}$*. We have*

$$\sum_n f_n z^n = \sum_n f_{n-1} z^n + \sum_n \left(\sum_{k<n} f_k \right) z^n + \sum_n \mathbb{1}[n > 0] z^n.$$

Note that

$$\sum_n \left(\sum_{k<n} f_k \right) z^n = \sum_k f_k \sum_{n>k} z^n = \sum_k f_k z^k \sum_{j>0} z^j = \sum_k f_k z^k \cdot \frac{1}{1-z}.$$

Then we have

$$F(z) = zF(z) + F(z) \cdot \frac{z}{1-z} + \frac{z}{1-z}$$

which yields that $F(z) = \frac{z}{1-3z+z^2}$*. Then we can expand the formula and calculate the coefficients in a similar way with the example of Fibonacci Sequence.*

2.2 Multiplication of Ordinary Generating Functions

Another useful rule to operate the generating functions is multiplication. That is

$$[z^n]F(z)G(z) = \sum_k f_k g_{n-k}.$$

Recall the problem of throwing k identical balls into n distinct bins where we allow no bins to contain more than t balls. Then the number of possible ways equals to

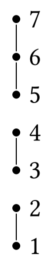
$$\begin{aligned} [z^k](1 + z + z^2 + \ldots + z^t)^n &= [z^k] \left(\frac{1 - z^{t+1}}{1-z} \right)^n \\ &= [z^k][\frac{1}{(1-z)^n} \cdot \left(1 - z^{t+1}\right)^n]. \end{aligned}$$

The coefficient of $\frac{1}{(1-z)^n} \cdot \left(1-z^{t+1}\right)^n$ can be computed using the multiplication rule based on the basic series we already know.

More generally, given n generating functions $F^{(1)}, F^{(2)}, \ldots, F^{(m)}$ where $F^{(i)} = \sum_n f_n^{(i)} z^n$, we can calculate the coefficients with m-fold convolution. That is,

$$[z^n]\left(F^{(1)}F^{(2)}\cdots F^{(m)}\right) = \sum_{i_1+i_2+\ldots+i_m=n} \prod_{j=1}^{m} f_{i_j}^{(j)}.$$

Consider a specific case where all $F^{(i)}$'s are equal, i.e., $F^{(i)} = G = \sum_n g_n z^n$ for all $i \in [m]$. In the problem of counting the spanning trees of a fan, we can first decide the connectivity of the vertices $\{1, 2, \ldots, n\}$. For example, there are $3 \times 2 \times 2 = 12$ spanning trees of a fan with 8 vertices if vertices $\{1, 2, \ldots, 7\}$ are connected as follows.



Then $f_n = \sum_{m>0} \sum_{i_1+i_2+\cdots+i_m=n,\ i_j>0} i_1 i_2 \cdots i_m$. Let $G(z) \triangleq \sum_{n>0} nz^n = \frac{z}{(1-z)^2}$. Then f_n equals to $[z^n]\sum_{m>0} G(z)^m$. Thus, the generating function with regard to f_n is

$$F(z) = \sum_{m>0} G(z)^m = \frac{G(z)}{1-G(z)} = \frac{z}{1-3z+z^2}.$$