[CS1961: Lecture 4] Generating Functions

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1 Ordinary Generating Functions

1.1 Useful Generating Functions

Recall that the ordinary generating function with regard to the sequence $\{f_k\}_{k\geqslant 0}$ is defined as $F(z) = \sum_{k\geqslant 0} f_k z^k = \sum_k f_k z^k$, where we let $f_k = 0$ for $k < 0$. Consider the following instances.

- $f_k \equiv 1$, $F(z) = \sum_{k\geqslant 0} z^k = \frac{1}{1-z}$.
- $f_k = c^k$, $F(z) = \sum_{k\geqslant 0} c^k z^k = \frac{1}{1-cz}$.
- $f_k = \binom{n}{k}$, $F(z) = \sum_{k\geqslant 0} \binom{n}{k} z^k = (1+z)^n$.
- $f_k = \left(\!\binom{n}{k}\!\right) = \binom{n+k-1}{k}$,

  $$F(z) = \sum_{k\geqslant 0} \left(\!\!\binom{n}{k}\!\!\right) z^k = (1+z+z^2\cdots)^n = \frac{1}{(1-z)^n}. \tag{1}$$

  Note that $\left(\!\binom{n}{k}\!\right)$ is the number of ways to throw k identical balls into n distinct bins. Let the power of z denote the number of balls. Then the balls in the first bin can be $1+z+z^2+\cdots$. Thus, the number of ways to throw k identical balls into n distinct bins equals to the coefficient of z^k in $(1+z+z^2\cdots)^n$. This gives a combinatorial proof to Equation (1).

- $f_n = p(n)$,[1]

  $$\begin{aligned} P(z) = \sum_{n\geqslant 0} p(n)z^n &= (1+z+z^2\cdots)(1+z^2+z^4\cdots)(1+z^3+z^6\cdots)\cdots \\ &= \frac{1}{1-z}\cdot\frac{1}{1-z^2}\cdot\frac{1}{1-z^3}\cdot\cdots = \prod_{i\geqslant 1}\frac{1}{1-z_i}. \end{aligned} \tag{2}$$

  The intuition behind Equation (2) is that the power of z^i in $(1+z^i+z^{2i}+z^{3i}\cdots)$ can be viewed as the number of i's in the partition of n. Then the number of total ways to partition n is exactly the coefficient of z^n in $\prod_{i\geqslant 1}\frac{1}{1-z_i}$.

[1] $p(n) \triangleq \sum_{k\geqslant 0} p(n,k)$ is the number of ways to partition n (the partition is not necessarily positive). For example, it is easy to see $p(4) = 5$ since $4 = 1+1+1+1 = 1+1+2 = 1+3 = 2+2$.

The following beautiful proof using generating function is due to Euler.

Example 1 (Odd partition and distinct partition) *Let o_n and d_n be the number of ways to partition n into odd numbers and distinct numbers respectively.*[2] *Then the generating function of $\{o_n\}$ is*

[2] For example, when $n = 6$, we have $o_n = 3$ which contains $1+1+\cdots+1$, $1+5$ and $3+3$ and $d_n = 3$ which contains $1+5$, $2+4$ and 6.

$$\begin{aligned}O(z) &= \sum_{n\geqslant 0} o_n z^n \\ &= (1+z+z^2\cdots)(1+z^3+z^6\cdots)(1+z^5+z^{10}\cdots)\cdots \\ &= \frac{1}{(1-z)(1-z^3)(1-z^5)\cdots}.\end{aligned}$$

Similarly, the generating function of $\{d_n\}$ *is*

$$\begin{aligned}D(z) = \sum_{n\geqslant 0} d(n)z^n &= (1+z)(1+z^2)(1+z^3)\cdots \\ &= \frac{1-z^2}{1-z}\cdot\frac{1-z^4}{1-z^2}\cdot\frac{1-z^6}{1-z^3}\cdot\ \cdots \\ &= \frac{1}{(1-z)(1-z^3)(1-z^5)\cdots}.\end{aligned}$$

Note that $D(z)$ *is the same with* $O(z)$*. Thus we have* $o_n = d_n$*, i.e., the number of odd partitions is equivalent to the number of distinct partitions.*

1.2 Operations on Generating Functions

Given $F(z) = \sum_{k\geqslant 0} f_k z^k$ and $G(z) = \sum_{k\geqslant 0} g_k z^k$, it holds that

- $[z^k](F+G) = f_k + g_k$;
- $[z^k](z^m F) = f_{k-m}$;
- $[z^k](cF) = cf_k$.

Let's see an application of these rules.

Example 2 (The Stirling number of the second kind) *Recall that the Stirling number of the second kind* $S(n,k)$ *is the number of ways to partition* $[n]$ *into exactly* k *parts. If we fix* k *and regard* $S(n,k)$ *as a sequence with respect to* n*, the ordinary generating function of* $S(n,k)$ *is* $S_k(z) = \sum_{n\geqslant 0} S(n,k)z^n$*. Specifically,* $S_0(z) = 1$*. Note that*

$$S(n,k) = S(n-1,k-1) + kS(n-1,k) + \mathbf{1}[n=k=0].$$

Multiplying both sides by z^n*, we have*

$$S(n,k)z^n = S(n-1,k-1)z^n + kS(n-1,k)z^n + \mathbf{1}[n=k=0]z^n.$$

Then we have $S_k(z) = zS_{k-1}(z) + kzS_k(z)$ *for* $k > 0$*. This yields that*

$$S_k(z) = \frac{z}{1-kz}S_{k-1}(z) = \frac{z}{1-kz}\cdot\frac{z}{1-(k-1)z}\cdot S_{k-2}(z) = \cdots = \frac{z^k}{\prod_{j=1}^{k}(1-jz)}.$$

Another useful operation on generating functions is convolution:

$$[z^n](F\cdot G) = \sum_k f_k g_{n-k}.$$

Example 3 (Catalan number) *Given the formula* $x_0 \cdot x_1 \cdot \dots \cdot x_n$*, let* c_n *be the number of ways to insert parentheses to specify the order of multiplication of* $x_0, x_1, \dots, x_n$.[3] *Then by enumerating the position of the last multiplication and splitting the formula into two smaller formulae, we know that* $c_n = \sum_{k=0}^{n-1} c_k \cdot c_{n-1-k}$*. Specifically,* $c_0 = 1$*. We can write the recurrence formula as*

$$c_n = \sum_{k=0}^{n-1} c_k \cdot c_{n-1-k} + \mathbf{1}[n = 0].$$

Multiply both side by z^n*:*

$$c_n z^n = \sum_{k=0}^{n-1} c_k \cdot c_{n-1-k} z^n + \mathbf{1}[n = 0] z^n.$$

Summing the formulae together, we have

$$C(z) = \sum_n c_n z^n = z \sum_n (\sum_{k=0}^{n-1} c_k c_{n-1-k} z^{n-1}) + 1 = zC(z)^2 + 1.$$

Then we can work out that $C(z) = \frac{1-\sqrt{1-4z}}{2z}$ *by solving the quadratic equation and using the condition* $C(0) = c_0 = 1$.

Note that by the generalized binomial theorem[4],

$$(1-4z)^{\frac{1}{2}} = \sum_{n \geqslant 0} \binom{\frac{1}{2}}{n} (-4z)^n = 1 + \sum_{n \geqslant 1} \binom{\frac{1}{2}}{n} (-4z)^n.$$

Thus,

$$\begin{aligned} C(z) &= \frac{1-(1-4z)^{\frac{1}{2}}}{2z} = -\frac{1}{2z} \sum_{n \geqslant 1} \binom{\frac{1}{2}}{n} (-4z)^n \\ &= 2 \sum_{n \geqslant 1} \binom{\frac{1}{2}}{n} (-4z)^{n-1} = 2 \sum_{n \geqslant 0} \binom{\frac{1}{2}}{n+1} (-4)^n z^n. \end{aligned}$$

This yields that

$$\begin{aligned} c_n = [z^n]C(z) &= 2 \binom{\frac{1}{2}}{n+1} \cdot (-4)^n \\ &= 2 \cdot \frac{\frac{1}{2}(\frac{1}{2}-1) \cdots (\frac{1}{2}-n)}{(n+1)!} \cdot (-2)^n \cdot 2^n \\ &= \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{(n+1)!} \cdot 2^n \\ &= \frac{1}{(n+1)!} 2^n \frac{(2n)!}{2^n n!} \\ &= \frac{(2n)!}{(n+1)!n!} = \frac{1}{n+1} \binom{2n}{n}. \end{aligned}$$

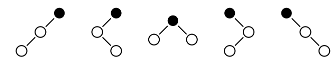
This is called the Catalan number.

[3] For example, $c_3 = 5$ since we have the following five ways:

$$((x_0x_1)x_2)x_3,$$
$$(x_0(x_1x_2))x_3,$$
$$(x_0x_1)(x_2x_3),$$
$$x_0((x_1x_2)x_3),$$
$$x_0(x_1(x_2x_3)).$$

[4] The generalized binomial theorem states that for an arbitrary number r, $(a+b)^r = \sum_{k \geqslant 0} \binom{r}{k} a^k b^{r-k}$ where $\binom{r}{k} \triangleq \frac{r(r-1)(r-2)\cdots(r-k+1)}{k!}$.

The Catalan number appears in many places of discrete mathematics, mainly due to its natural recurrence pattern.



Example 4 (Distinct binary trees) *Another example is counting the number of distinct binary trees with* n *vertices. Let* $T(n)$ *be the number of such binary trees. For example, when* $n = 3$*, we have* $T(3) = 5$.

Consider the structure of the two subtrees of the root. We have $T(n) = \sum_{k=0}^{n-1} T(k)T(n-1-k)$ *and* $T(0) = 1$*. Similarly, we have* $T(n) = c_n = \frac{1}{n+1}\binom{2n}{n}$.

2 *Exponential Generating Functions*

Consider a sequence of numbers $\{f_n\}_{n \geqslant 0}$ that $f_n = nf_{n-1}$ and $f_0 = 1$. It is easy to see that $f_n = n!$. However, we can take a little detour and solve this problem from the perspective of generating functions.

We write the recursion as $f_n = nf_{n-1} + \mathbf{1}[n = 0]$. Multiply both sides by z^n. We have

$$f_n z^n = nf_{n-1}z^n + \mathbf{1}[n = 0]z^n.$$

Note that $F'(z) = \sum_n nf_n z^{n-1}$. Then

$$z^2F' = \sum_n nf_{n-1}z^n - \sum_n f_{n-1}z^n = \sum_n nf_{n-1}z^n - zF.$$

Thus, $F(z)$ satisfies $z^2F' + (z-1)F + 1 = 0$. There is no closed form solution of this differential equation. [5] The ordinary generating function does not work well on this sequence. To work out this problem, we introduce the exponential generating functions.

[5] It can be solved using the technique of hypergeometric series. See the Chapter 5 of *Concrete Mathematics*.

Definition 1 (Exponential generating functions) *Given a sequence* $\{f_k\}_{k \geqslant 0}$*, the exponential generating function with regard to* $\{f_k\}_{k \geqslant 0}$ *is*

$$\widehat{F}(z) \triangleq \sum_{k \geqslant 0} f_k \frac{z^k}{k!}$$

.

For example, when

- $f_k = k!$, $\widehat{F}(z) = \sum_{k \geqslant 0} z^k = \frac{1}{1-z}$;
- $f_k = 1$, $\widehat{F}(z) = \sum_{k \geqslant 0} \frac{z^k}{k!} = e^z$.

Back to the above problem, dividing both sides by $n!$, we have

$$\frac{f_n z^n}{n!} = \frac{f_{n-1}z^n}{(n-1)!} + \mathbf{1}[n = 0] \cdot \frac{z^n}{n!}.$$

Take sum for all n. We have

$$\widehat{F}(z) = z\widehat{F}(z) + 1.$$

This yields that $\widehat{F}(z) = \frac{1}{1-z} = 1 + z + z^2 \cdots$. Thus we have $f_n = n!$.

Consider another sequence $\{f_n\}_{n\geqslant 0}$ that $f_n = nf_{n-1} + 1$ and $f_0 = 1$. Then we have

$$f_n \frac{z^n}{n!} = f_{n-1}\frac{z^n}{(n-1)!} + \frac{z^n}{n!}$$

and sequentially $\widehat{F}(z) = z\widehat{F}(z) + e^z$. Therefore,

$$\widehat{F}(z) = \frac{e^z}{1-z} = (1 + z + z^2 + \cdots)(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \cdots),$$

and $f_n = [z^n]\widehat{F}(z)n! = n! \sum_{k=0}^{n} \frac{1}{k!}$.

There is also a multiplication rule for exponential generating functions. Let $\widehat{F}(z) = \sum_{n\geqslant 0} f_n \frac{z^n}{n!}$ and similarly define $\widehat{G}$. Let $\widehat{H}(z) = \sum_{n\geqslant 0} h_n \frac{z^n}{n!} = \widehat{F}(z) \cdot \widehat{G}(z)$. Then

$$h_n = n! \sum_k \frac{f_k}{k!} \cdot \frac{g_{n-k}}{(n-k)!} = \sum_k \binom{n}{k} f_k g_{n-k}.$$

Consider the problem of counting spanning trees. Let t_n be the number of spanning trees in an n vertices graph. For example, when $n = 3$, we have $t_3 = 3$. A natural idea is to use induction. Imagine that we delete the first vertex and count the number of connected blocks in the remaining graph. Once we fix the m connected blocks, the problem is reduced to constructing m smaller spanning trees and connecting vertex 1 with them. Thus,

$$t_n = \sum_{m>0} \frac{1}{m!} \sum_{k_1+k_2+\cdots k_m = n-1} \binom{n-1}{k_1, k_2 \ldots k_m} k_1 k_2 \cdots k_m t_{k_1} t_{k_2} \cdots t_{k_m}$$

where $k_i \geqslant 1$ and

$$\binom{n-1}{k_1, k_2, \ldots, k_m} = \binom{n-1}{k_1}\binom{n-1-k_1}{k_2}\binom{n-1-k_1-k_2}{k_3}\cdots = \frac{(n-1)!}{k_1!k_2!\cdots k_m!}.$$

By direct calculation, we have

$$t_n = \sum_{m>0} \frac{1}{m!} \sum_{k_1+k_2+\cdots k_m = n-1} (n-1)! \frac{t_{k_1}}{(k_1-1)!} \frac{t_{k_2}}{(k_2-1)!} \cdots \frac{t_{k_m}}{(k_m-1)!}.$$

Let $u_n = nt_n$. Then

$$\frac{u_n}{n!} = \sum_{m>0} \frac{1}{m!} \sum_{k_1+k_2+\cdot k_m = n-1} \prod_{i=1}^{m} \frac{u_{k_i}}{k_i!}.$$

This indicates that

$$[z^n]\widehat{U}(z) = [z^{n-1}] \sum_{m\geqslant 0} \frac{1}{m!} \widehat{U}^m(z) = [z^{n-1}]e^{\widehat{U}(z)} = [z^n]ze^{\widehat{U}(z)}.$$

Therefore, $\widehat{U}$ satisfies the differential equation $\widehat{U} = ze^{\widehat{U}}$. By the Lagrange inversion theorem[6], we can work out that $[z^n]\widehat{U} = \frac{k^{k-2}}{(k-1)!}$ and correspondingly $t_n = \frac{u_n}{n} = n^{n-2}$. This expression is called Cayley's formula.

[6] A combinatorial reasoning on this is given in the Chapter 7 of *Concrete Mathematics*.