[CS1961: Lecture 5] Partially Ordered Set

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1 Partially Ordered Set

1.1 Definition

Definition 1 (Partially ordered set) *A partially ordered set (or poset for short)* $(P, \leqslant)$ *is a set* P *equipped with a partial order* $\leqslant$*. Recall that we say* $\leqslant \subseteq P \times P$ *is a partial set if it satisfies the following three properties:*

- *reflexivity:* $\forall a \in P$*,* $a \leqslant a$*;*
- *antisymmetry:* $\forall a, b \in P$ *that* $a \leqslant b$ *and* $b \leqslant a$*, we have* $a = b$*;*
- *transitivity:* $\forall a, b, c \in P$ *that* $a \leqslant b$ *and* $b \leqslant c$*, we have* $a \leqslant c$*.*

Totally ordered set is a special kind of poset, which further satisfies $\forall a, b \in P$, either $a \leqslant b$ or $b \leqslant a$ is true.

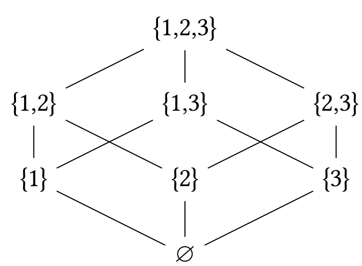
We can further define the notation $<$ as $\forall a, b \in P$, $a < b$ iff $a \leqslant b \wedge a \neq b$.

For example, given a universe U, $\left(2^U, \subseteq\right)$ is a poset. We can correspondingly define $\subsetneq$ on 2^U.

Divisibility is also a partial order. For $a, b \in \mathbb{Z}$, define the relation $\leqslant$ as $a \leqslant b$ iff $a|b$. Then $(\mathbb{Z}, \leqslant)$ is a poset.

1.2 Hasse Diagram

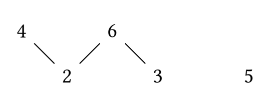
Hasse diagram is offen used to represent a partially ordered set. For $a, b \in P$, there is an edge (a, b) in the diagram iff $a < b$ and there does not exist such $c \in P$ that $a < c \wedge c < b$. Every poset can be represented as a Hasse diagram. For example, let $P = 2^{\{1,2,3\}}$. The poset $\left(2^{\{1,2,3\}}, \subseteq\right)$ can be represented as the following Hasse diagram.



An element x in a poset $\{P, \subseteq\}$ is called a *maximal* element if there does not exist some $y \in P$ satisfying $x < y$. The *maximum* element is the element x such that $\forall y \in P, y \leqslant x$. The notions of minimal element and minimum element are similarly defined.

A poset may not have maximum and minimum elements. There might even be isolated elements in the Hasse diagram.[1]

[1] An example is the divisibility relation on $P = \{2, 3, 4, 5, 6\}$:



Definition 2 (Chain) *A set* $C \subseteq P$ *is called a chain of the poset* $(P, \leqslant)$ *if*

$$\forall x, y \in C,\ x \leqslant y \vee y \leqslant x.$$

Intuitively, the elements in a chain form a (probably inconsecutive) path in the Hasse diagram. The maximal chain is such that adding any other elements will make it no longer a chain. The maximum chain is the chain with the maximum number of elements. For example, in Figure 1, the elements on the green path forms a maximal chain while the set of elements on the blue path is a maximum chain. The height of the poset $(P, \leqslant)$ is defined as the size of the maximum chain, which is 5 in Figure 1.

Definition 3 (Anti-chain) *A set* $T \subseteq P$ *is called an anti-chain of the poset* $(P, \leqslant)$ *if*

$$\forall x, y \in T,\ x \not\leqslant y \wedge y \not\leqslant x.$$

The elements in an anti-chain can not be compared with each other. Similarly, we have the notion of maximal and maximum anti-chain. The width of the poset $(P, \leqslant)$ is defined as the size of the maximum anti-chain. For example, the red vertices in Figure 1 form a maximal anti-chain and a maximum anti-chain at the same time. The width of this poset is 3.

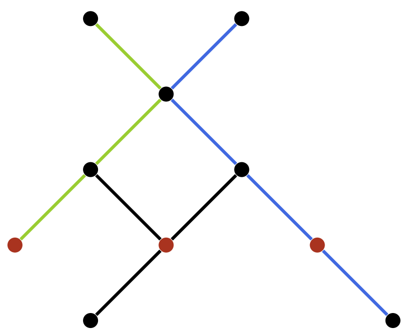
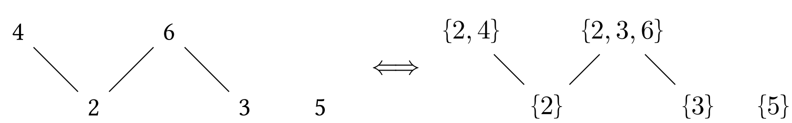


Figure 1: A Hasse Diagram

Actually, set inclusion is a canonical partial order (at least for finite posets). Every finite poset can be described using set inclusion by letting the parent be the union of itself and its children in the Hasse diagram (defined recursively from the leaves). For example, the division on $P = \{2, 3, 4, 5, 6\}$ can be regarded as the set inclusion on $P' = \{\{2\}, \{3\}, \{5\}, \{2, 4\}, \{2, 3, 6\}\}$.



2 *Chain Cover and Anti-Chain Cover*

Let $(P, \leqslant)$ be a poset. A *chain cover* of P is a collection of *disjoint* chains $\mathcal{C} = \{C_1, C_2, \ldots, C_m\}$ whose union is P. That is, $\bigcup \mathcal{C} = P$ and for any

$i, j \in [m]$, if $i \neq j$ then $C_i \cap C_j = \varnothing$. Similarly, we can define the notion of anti-chain cover $\mathcal{A} = \{A_1, \ldots, A_m\}$ as a collection of disjoint anti-chains whose union is P.

In this section, we will establish two *duality-type* theorems to connect the size of the minimum chain cover with the width of the poset, and connect the size of the minimum anti-chain cover with the height of the poset. Interestingly, these two facts are equivalent to many famous min-max theorem in combinatorics and there are in fact the LP duality theorem in disguise.

2.1 Anti-Chain Cover

Theorem 4 *The size of the minimum anti-chain cover of* $(P, \leqslant)$ *equals to the height of* $(P, \leqslant)$.

Proof. Let r be the height of the poset. It is clear that at least r anti-chains is needed. Then we construct a cover that contains exactly r anti-chains.

First we take the maximum element in each chain with size r and these elements form an anti-chain. Delete these elements and recursively construct anti-chains in the remaining poset with height $r - 1$. Finally we have r disjoint anti-chains which covers the original poset. □

2.2 Chain Cover

Theorem 5 (Dilworth's theorem) *The size of the minimum chain cover of* $(P, \leqslant)$ *equals to the width of* $(P, \leqslant)$.

Proof. Let w be the width of the poset. It is clear that at least w chains are needed. Then we prove that w chains are enough to cover the poset by induction.

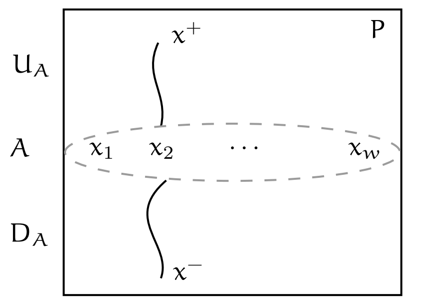
It is obvious when $|P| = 1$ and $|P| = 2$. Let $C \subseteq P$ be the maximum chain in P. Let $P' = P \setminus C$.

If the width of P' is reduced to $w - 1$, then P' can be covered with $w - 1$ chains by the induction hypothesis. This completes the proof in this case.

If the width of P' is still w, let $P'' = P \setminus \{x^+, x^-\}$ where x^+ and x^- are the maximum and minimum elements in C. Then there exists an anti-chain $A = \{x_1, x_2, \ldots, x_w\}$ in P'' and A divides P into three parts $A \sqcup U_A \sqcup D_A$ where $U_A \triangleq \{y \in P \mid \exists x \in A, x < y\}$ and $D_A \triangleq \{y \in P \mid \exists x \in A, y < x\}$.

Note that x^+ must be in U_A and x^- must be in D_A. Therefore we have $\left|U_A \cup A\right| < |P|$ and $\left|D_A \cup A\right| < |P|$. Applying the induction hypothesis on $U_A \cup A$, we can find w disjoint chains $C_{u_1}, C_{u_2}, \ldots, C_{u_w}$ to cover $U_A \cup A$ where C_{u_i} is the chain containing x_i. Similarly, we have $D_A \cup A = C_{d_1} \sqcup C_{d_2} \sqcup \cdots \sqcup C_{d_w}$. Then we can get w longer chains in P by connecting each C_{u_i} with C_{d_i}.

□



3 Min-max Theorems

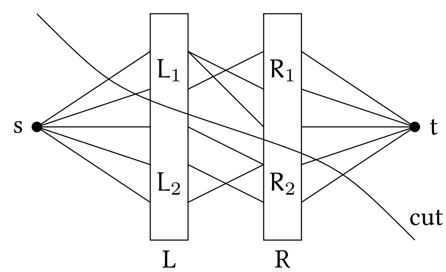
In this section, we will introduce *Hall marriage theorem* and *König's theorem*. We will show that they are all equivalent to Dilworth's theorem.

3.1 Hall's Marriage Condition

Theorem 6 (Hall's marriage condition) *Let* $G = (L, R, E)$ *be a bipartite graph where* $|L| = |R| = n$*. Then there exists a perfect matching in* G *iff* $\forall S \subseteq L, |N(S)| \geqslant |S|$. [2]

[2] $N(S) \triangleq \{y \in R \mid \exists x \in S, (x, y) \in E\}$ is the set of neighbors of the vertices in S.

Proof. (using max-flow min-cut theorem) The necessity is obvious. Then we only need to prove that if $\forall S \subseteq L, |N(S)| \geqslant |S|$, there exists a perfect matching in G.



As shown above, we connect L and R with a source s and a sink t respectively. Let the edges (s, l_i) and (r_j, t) have capacity 1 for every $l_i \in L$ and $r_j \in R$. Let the edges between L and R have infinite capacity. If the maximum value of the flow from s to t equals to n, there must exist a perfect matching in G.

Note that the maximum value of the flow equals to the minimum capacity of a cut and the min-cut capacity must be no larger than n. Therefore, we only need to prove that if the Hall's marriage condition holds, i.e., $\forall S \subseteq L, |N(S)| \geqslant |S|$, the min-cut capacity is no less than n.

We claim that the minimum cut cuts no edges between L and R since otherwise the cut capacity will be infinite. Let C be the min-cut capacity.

Then we have

$$C = |L_1| + |R_2| \geqslant |L_1| + |N(L_2)| \geqslant |L_1| + |L_2| = n.$$

This yields that the max-flow value equals to n and there exists a perfect matching in G. □

There is another way to prove Theorem 6 using Dilworth's theorem.

Proof of Theorem 6. (using Dilworth's theorem) Let $P = L \cup R$. Construct a partial order that $l_i \leqslant r_j$ iff $(l_i, r_j) \in E$. Then G is the Hasse diagram of $(P, \leqslant)$.

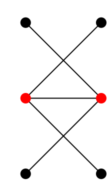
We claim that $\{r_1, r_2, \ldots, r_n\}$ is a maximum anti-chain. Assume in contradiction that the maximum anti-chain is $L_1 \cup R_2$ where $L_1 \subseteq L$ and $R_2 \subseteq R$ that $|L_1| + |R_2| > n$. Then by Hall's marriage condition, we can replace L_1 with $N(L_1)$ to get another anti-chain whose size is no less than $L_1 \cup R_2$. This is in contradiction with the assumption that $|L_1| + |R_2| > n$. Thus, the width of $(P, \leqslant)$ is n. Applying Dilworth's theorem, there exists n disjoint chains which can cover P. These n chains exactly form a perfect matching in G.

□

3.2 König's Theorem

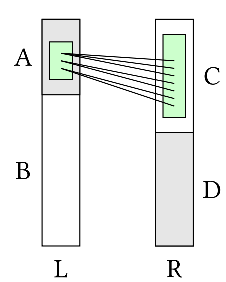
Theorem 7 (König's Theorem) *In a bipartite graph $G = (L, R, E)$, the size of the maximum matching equals to the size of the minimum vertex cover.*[3]

[3] A vertex cover is a set of vertices that includes at least one endpoint of every edge.



For example, in the graph above, the size of maximum matching = the size of minimum vertex cover (the set of red vertices) = 2.

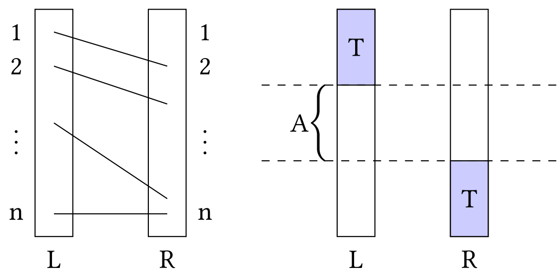
Proof of König's theorem using Hall's marriage condition It is easy to know that the size of an vertex cover is always no less than the size of any matching. Then we prove that there exists a matching that has the same size with the minimum vertex cover.



Assume that $A \cup D$ is a minimum vertex cover where $A \subseteq L$ and $D \subseteq R$. Then as shown above, for every $S \subseteq A$, $|N(S) \cap C| \geqslant |S|$. That is, A satisfies Hall's marriage condition in the subgraph $G[A \cup C]$ since otherwise we can replace some $S \subseteq A$ with its smaller neighbor set and get a smaller vertex cover. Thus, there exists a matching of size $|A|$ in $G[A \cup C]$. Similarly, we can find a matching of size $|D|$ in $G[B \cup D]$. This forms a matching which has the same size with the minimum vertex cover. $\square$

König's theorem can be used to prove Dilworth's theorem.

Proof of Dilworth's theorem using König's theorem. Let $V = \{1, 2, \ldots, n\}$ and $\leqslant$ is a partial order on V. Then we construct a bipartite graph $G = (L, R, E)$ that $L = R = V$ and $(i, j) \in E$ iff $i \leqslant j$.



Note that a matching M in G induces a disjoint chain cover $P = \{C_1, C_2, \ldots, C_m\}$ where $C_k = \left\{x_1^{(k)}, x_2^{(k)}, \ldots, x_{n_k}^{(k)}\right\}$ that $\left(x_i^{(k)}, x_{i+1}^{(k)}\right) \in M$. Each C_k contains $n_k - 1$ edges in M. Therefore, $\sum_{k=1}^{m}(n_k - 1) = |M|$. This yields that $n - m = |M|$ since $\sum_{k=1}^{m} n_k = n$.

Moreover, we can construct an anti-chain in V based on a vertex cover S in G. Let $T \subseteq S$ be the same with S but without duplication[4]. Let $A = V \setminus T$. Then A is an anti-chain of $(V, \leqslant)$.

By Theorem 7, if we choose the maximum matching M^* and the minimum vertex cover S^*, $|M^*| = |S^*|$. Then

$$|A| = n - |T| \geqslant n - |S^*| = n - |M^*|.$$

Since $n - |M^*| = m$, we have $|A| \geqslant m$. That is, we can cover the poset with no more than $|A|$ chains where $|A|$ is no larger than the width of $(V, \leqslant)$. $\square$

[4] For example, if $S = \{1, 1, 2, 3\}$, then $T = \{1, 2, 3\}$.

Remark 1 *We have the following relationship among Dilworth's theorem, Hall's marriage condition and König's theorem. This indicates that these three theorems are equivalent. In fact, as shown before, we can derive all of them from the maxflow mincut theorem, which is a consequence of the strong duality theorem for linear programs.*

Dilworth's theorem

$\Swarrow \quad \Nwarrow$

Hall's marriage condition $\Longrightarrow$ König's theorem