Do you feel lucky?

CS1212 Introduction to Theoretical Computer Science
Lecture 9-11

Kuan Yang

水源社区

一种图鉴抽卡类游戏的概率问题

水源广场　谈笑风生

R我所愿　10 月 16 日

刚才摸鱼摸到把池塘里的鱼都摸光了，给自己找了条大鱼摸：

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An introduction to probability and randomized algorithms

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- $O(d^2)$ multiplication or $O(d \log d)$ with Fourier transform

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- Select $c_1, c_2 \in S$ independently, then $\Pr[\forall i, f(c_i) = g(c_i)] \leq 1/100^2$
- $O(d \log(1/\varepsilon))$ times multiplication if allow mistake probability ε

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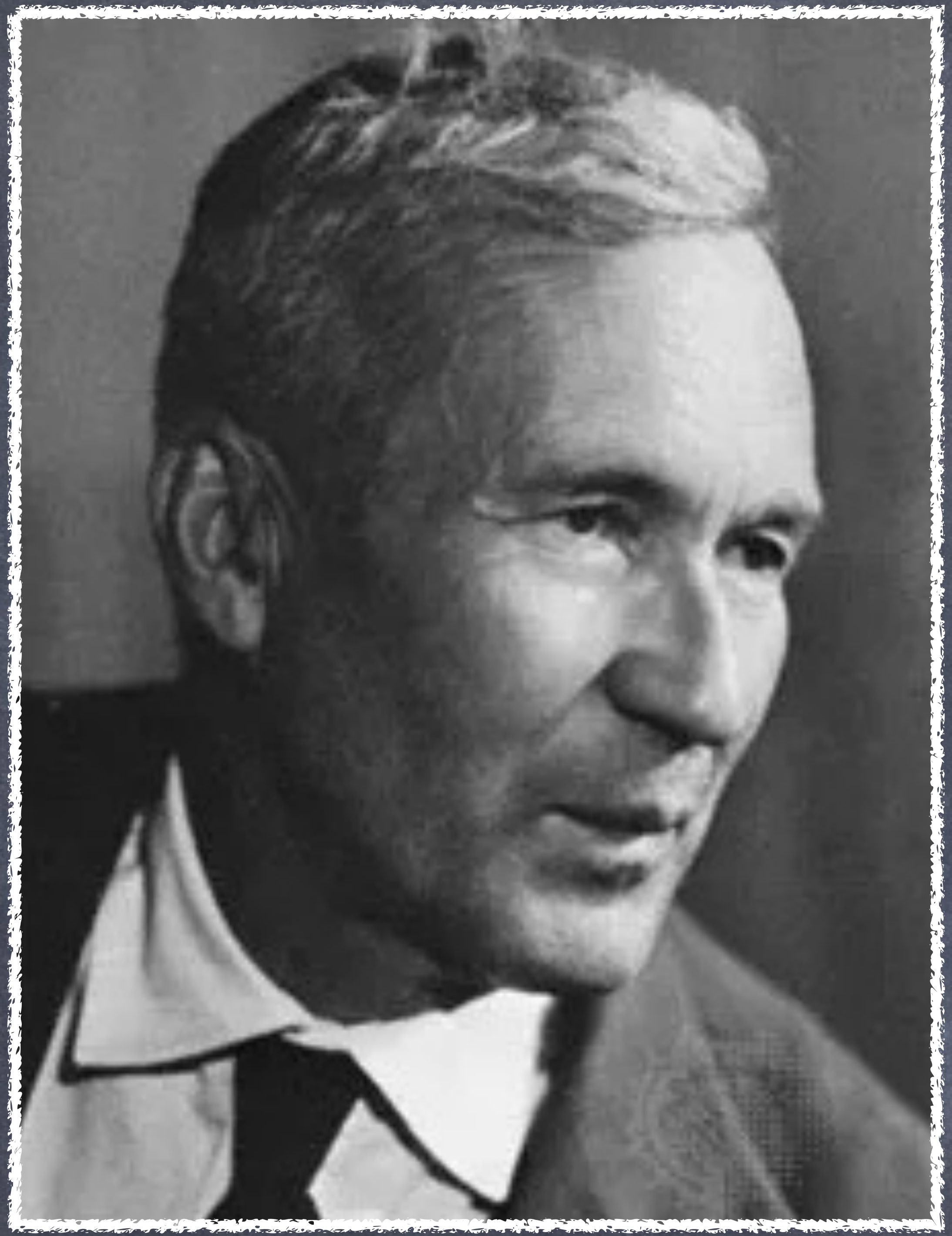
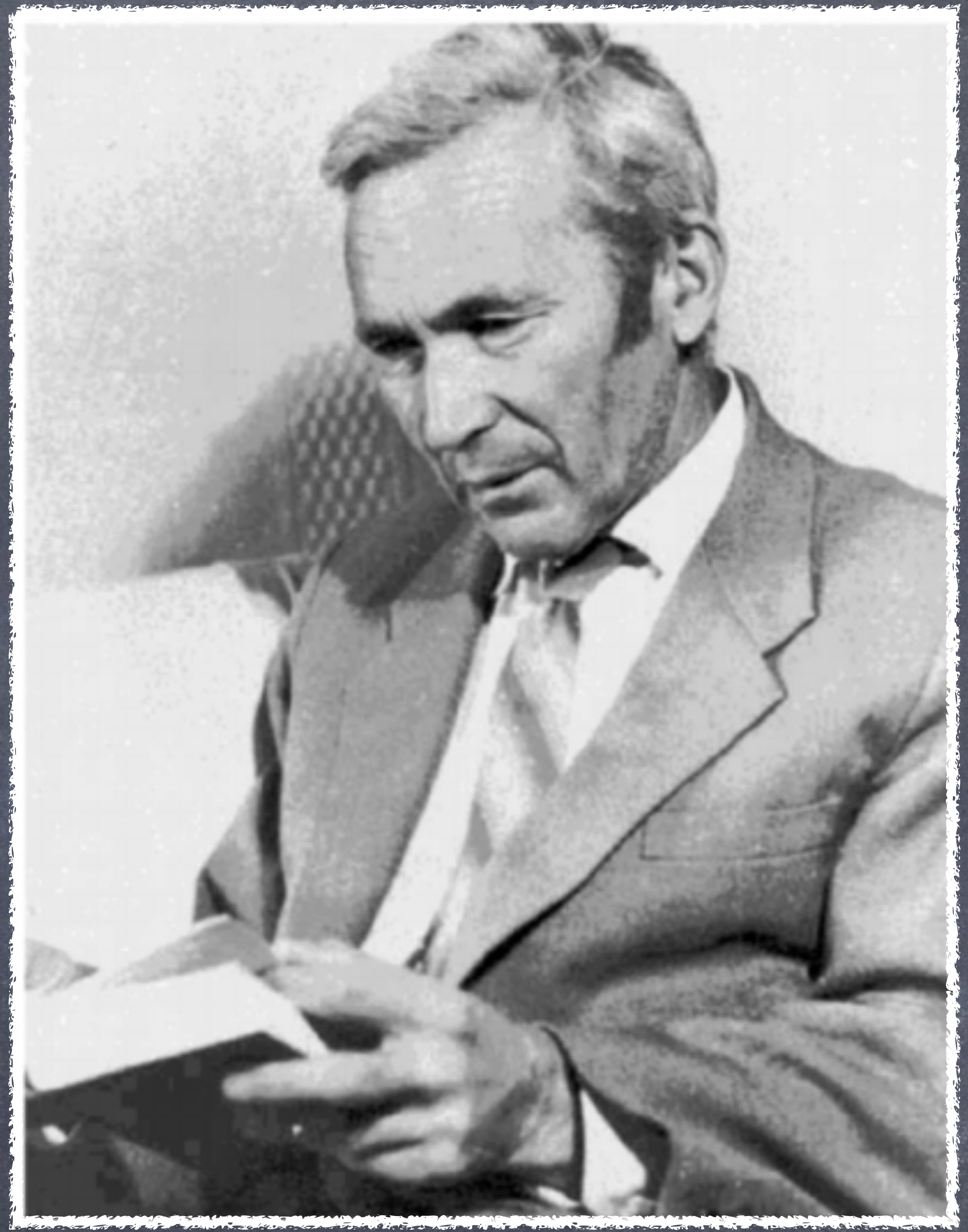
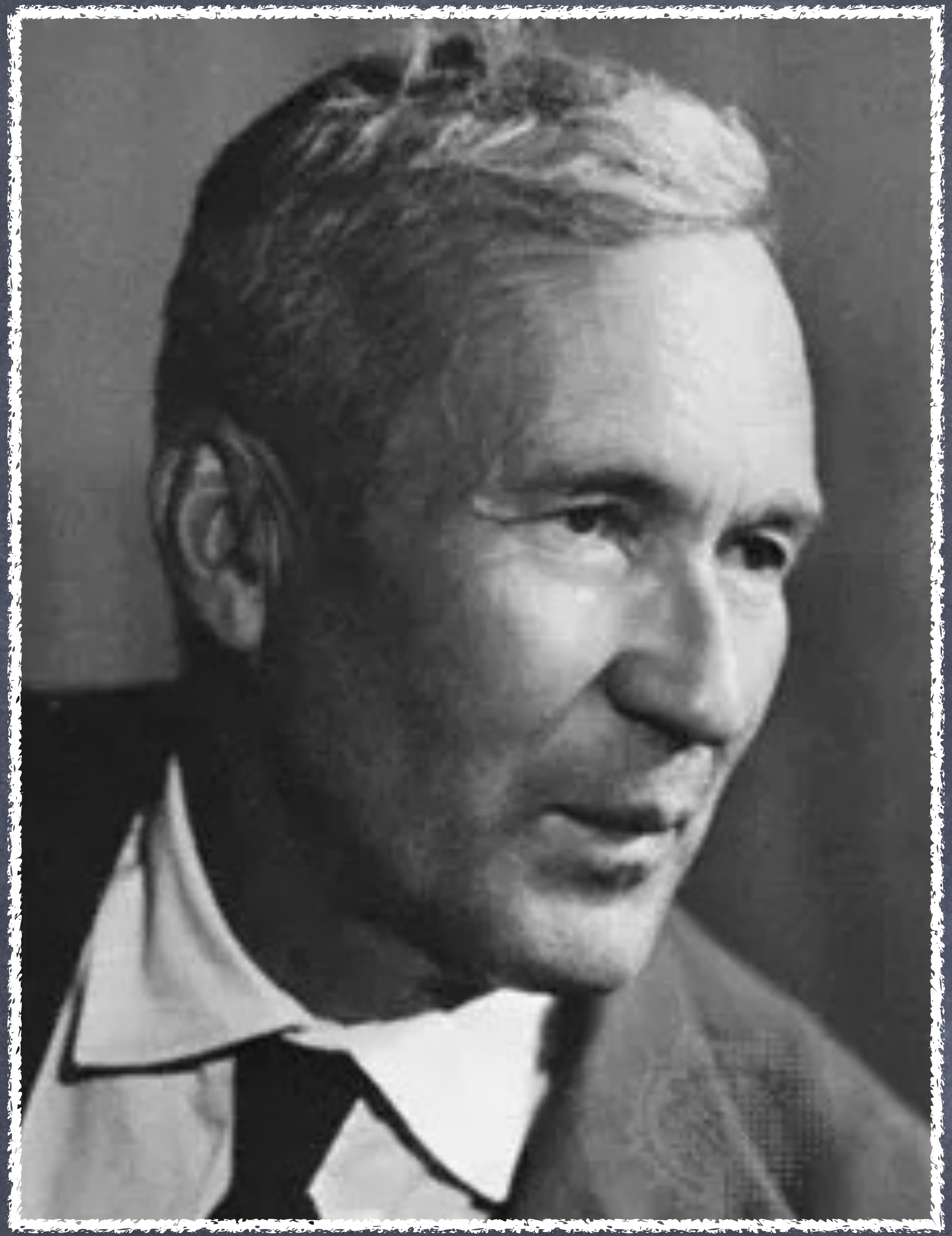
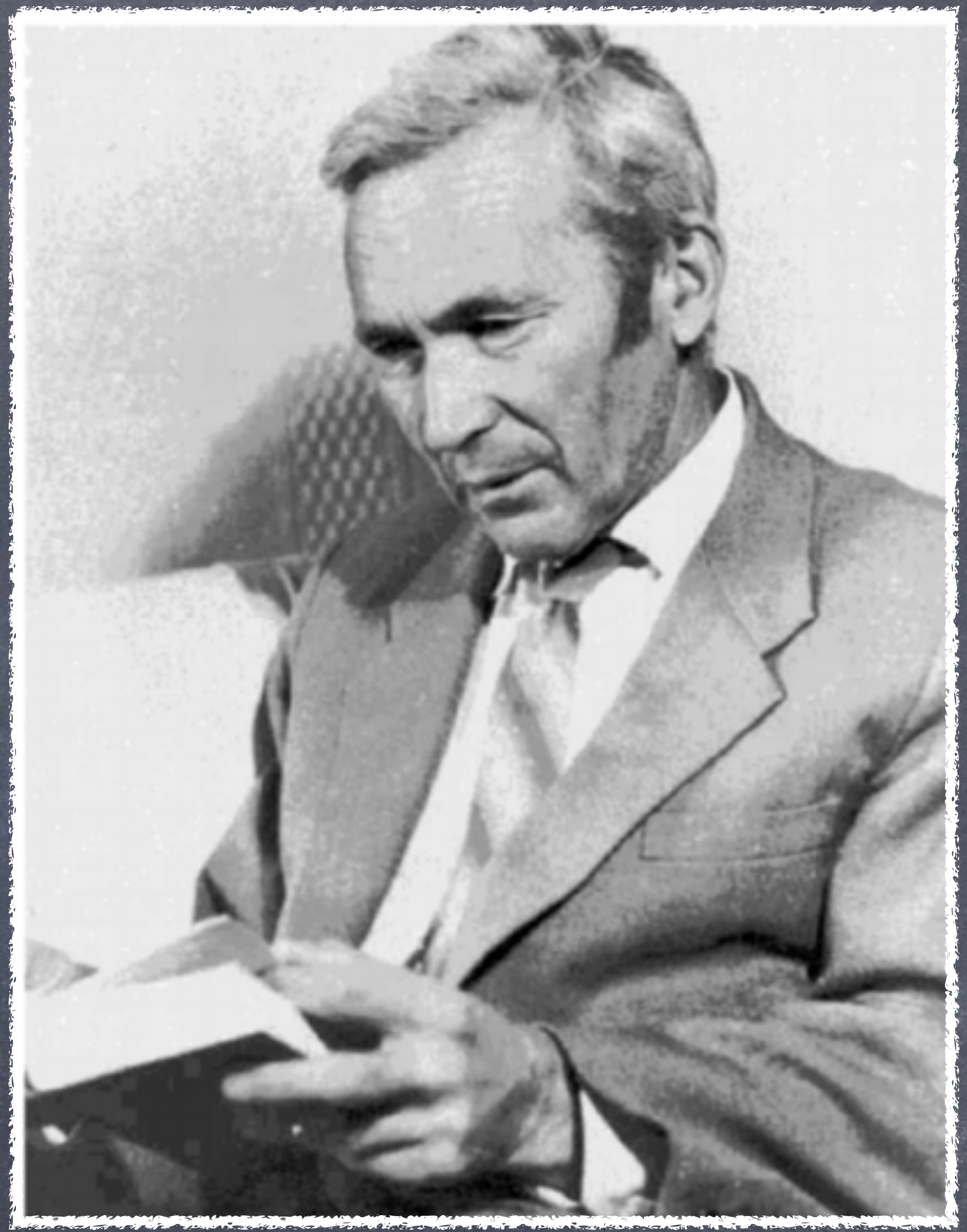
- There are two boxes having x and $\lfloor x/2 \rfloor$ coins respectively. Open a box and find 100 coins. $\mathbb{E}[\text{coins in another}] = 125$?





Andrey Nikolaevich Kolmogorov

(1903.4.25 – 1987.10.20)

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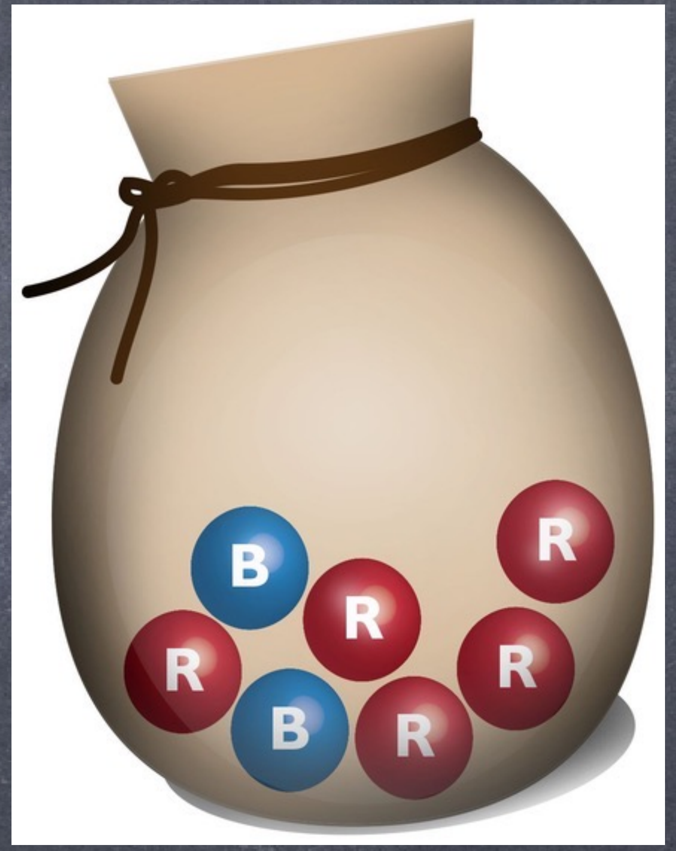

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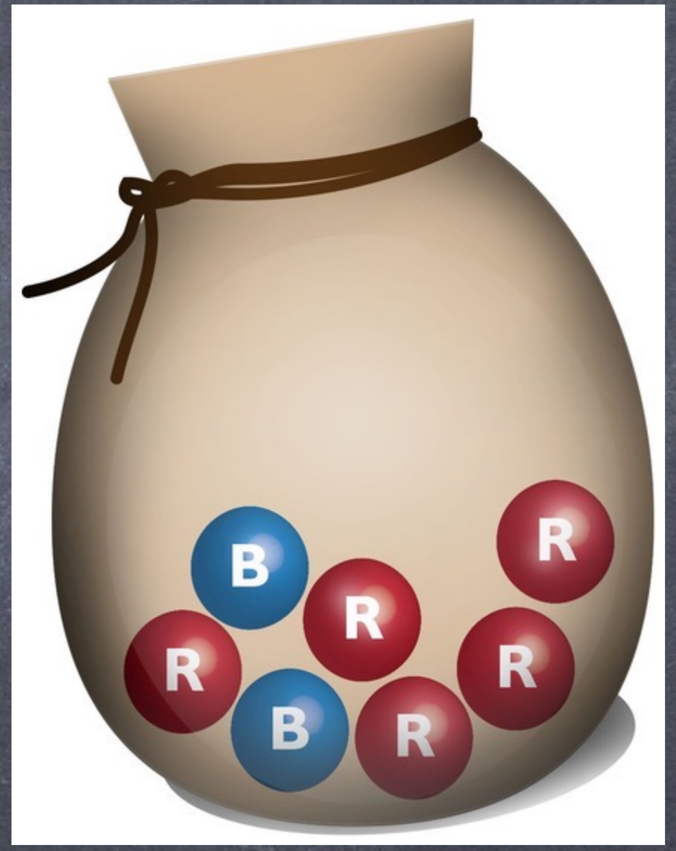

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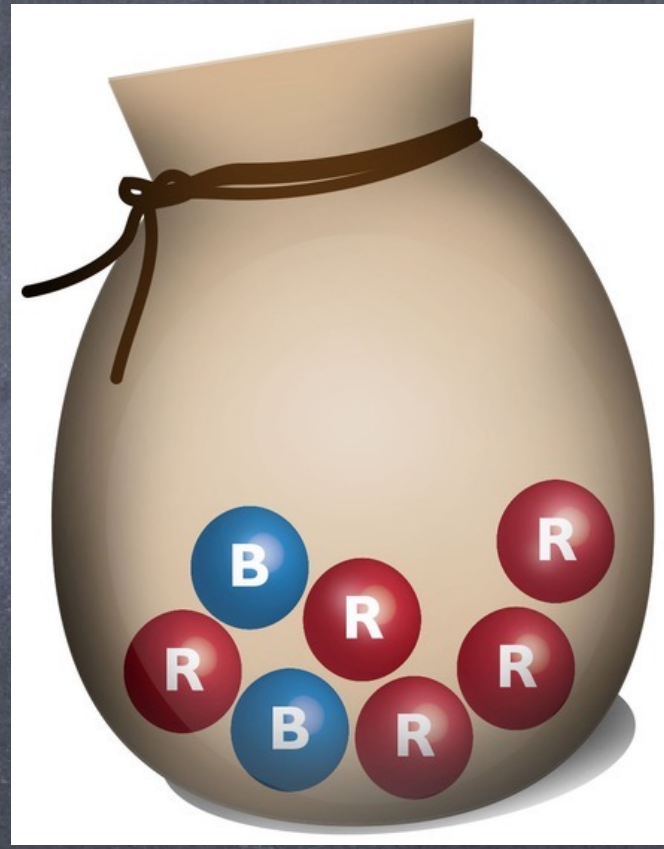

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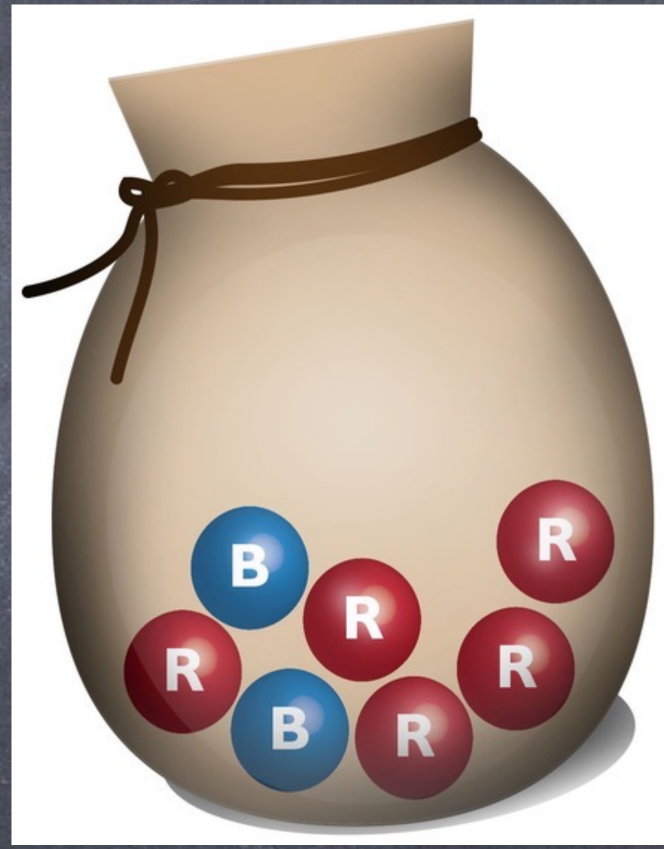

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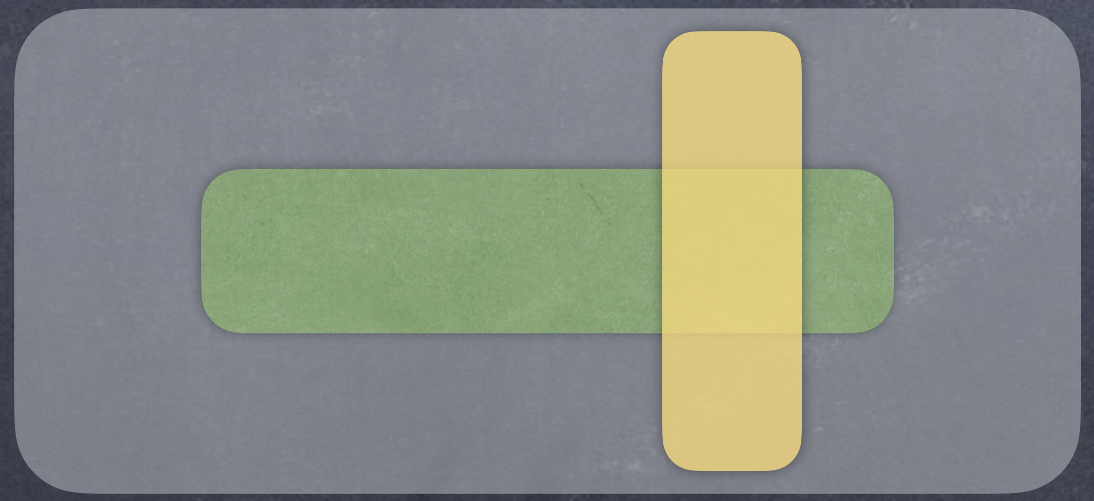

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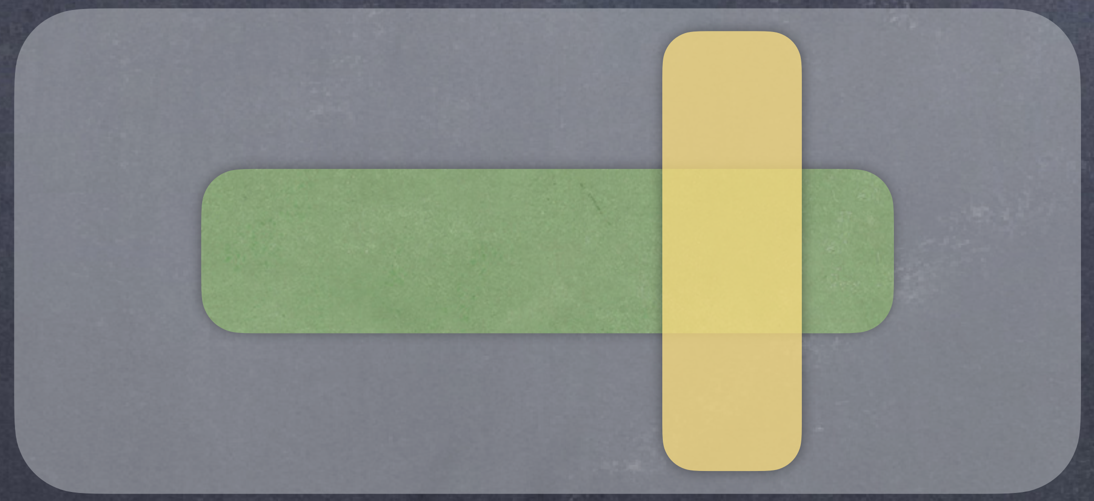

- Disjoint events are highly dependent!

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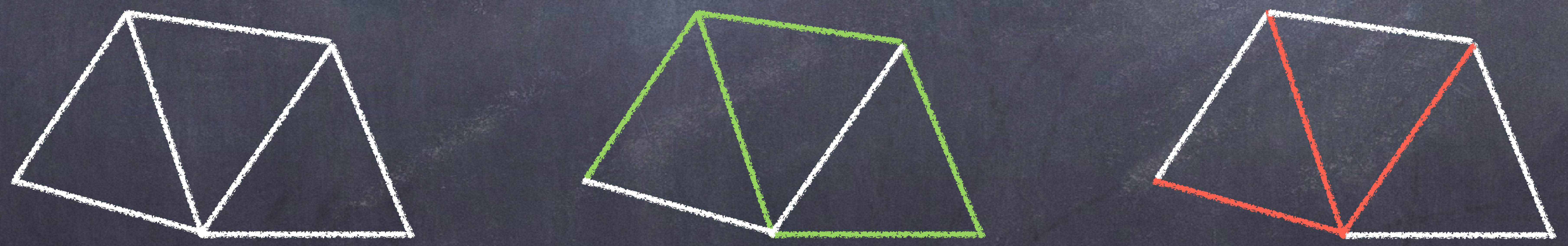

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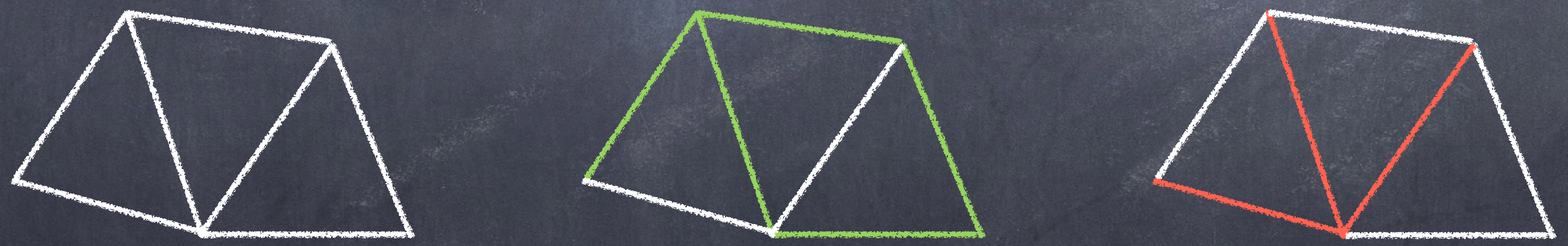

- Min-cut: a cut of the minimum size

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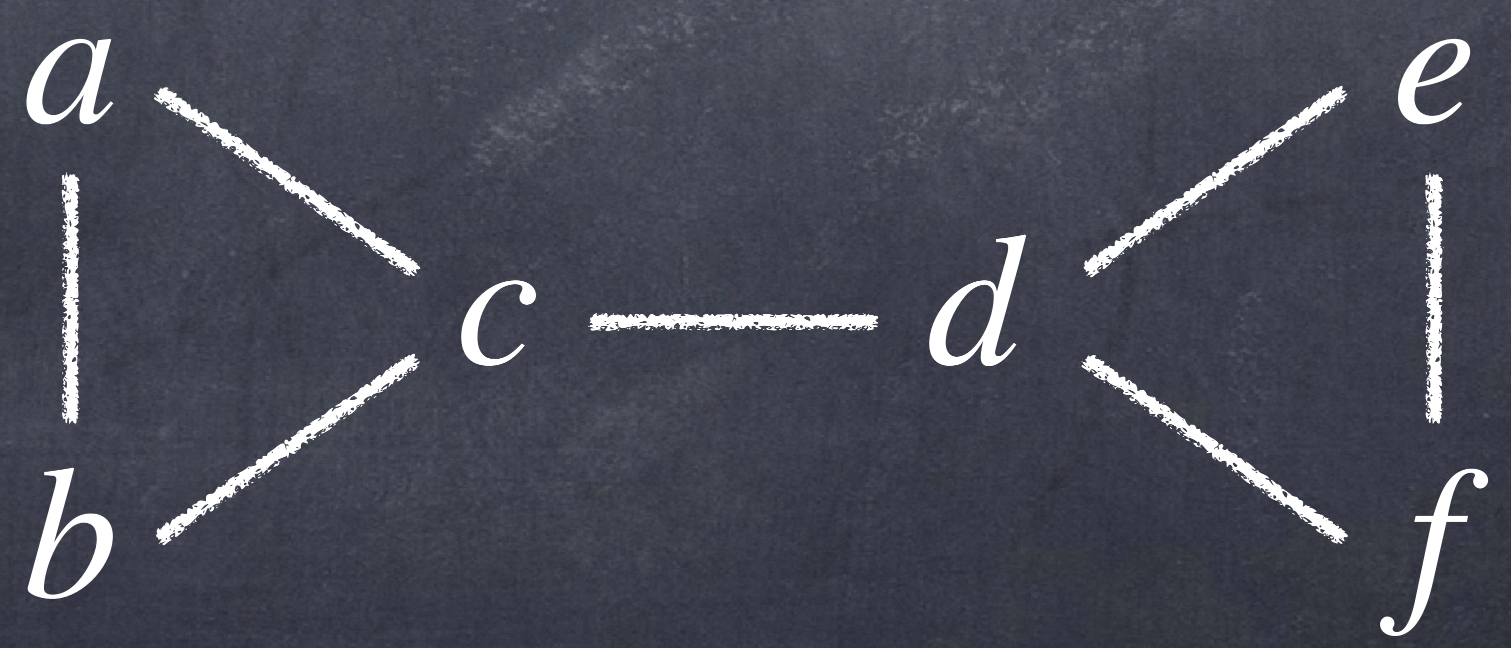

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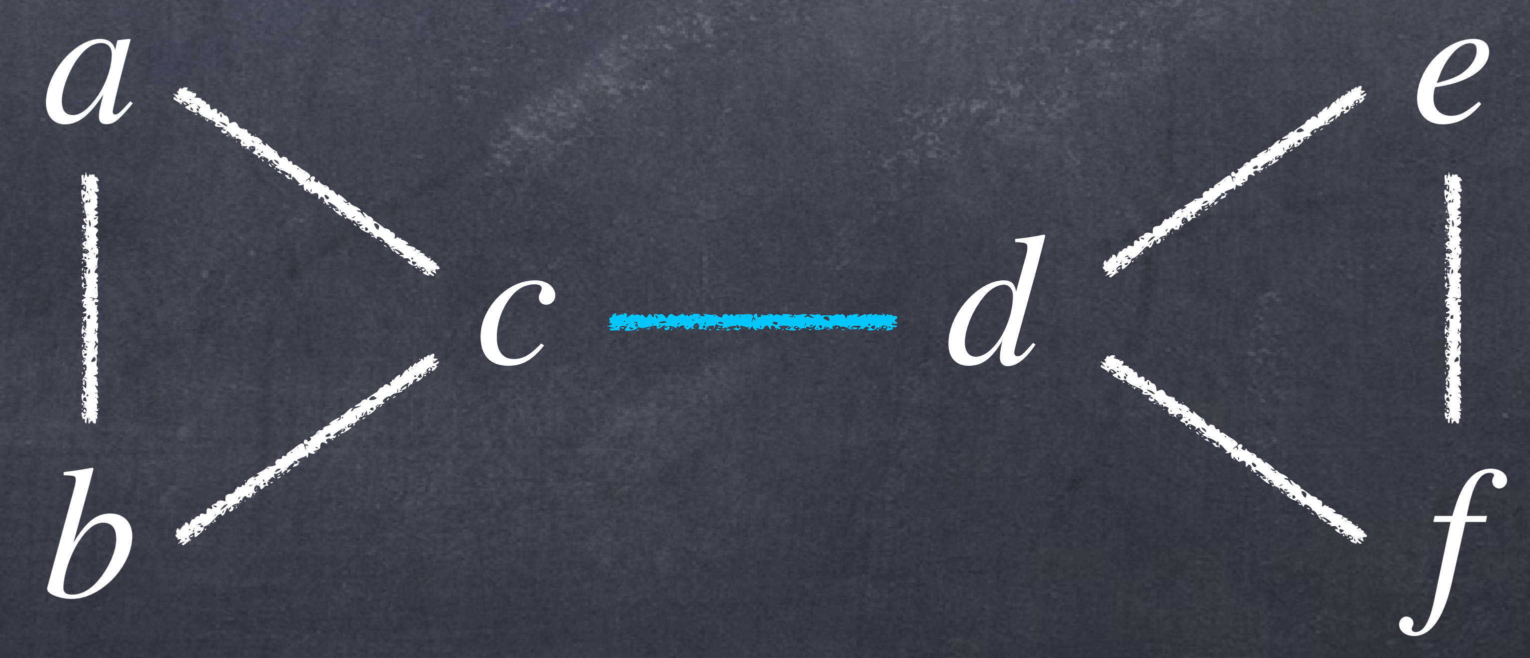

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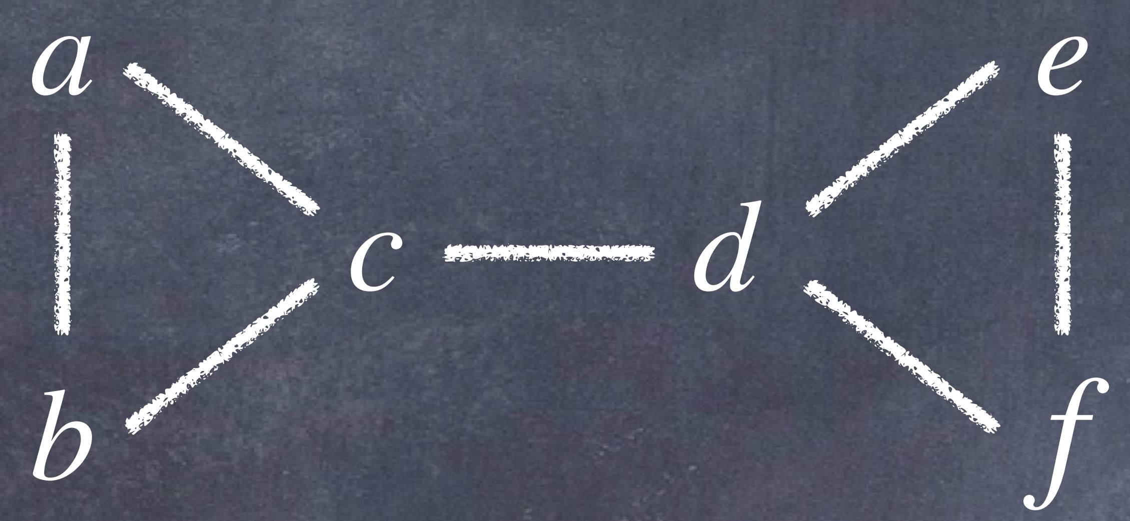

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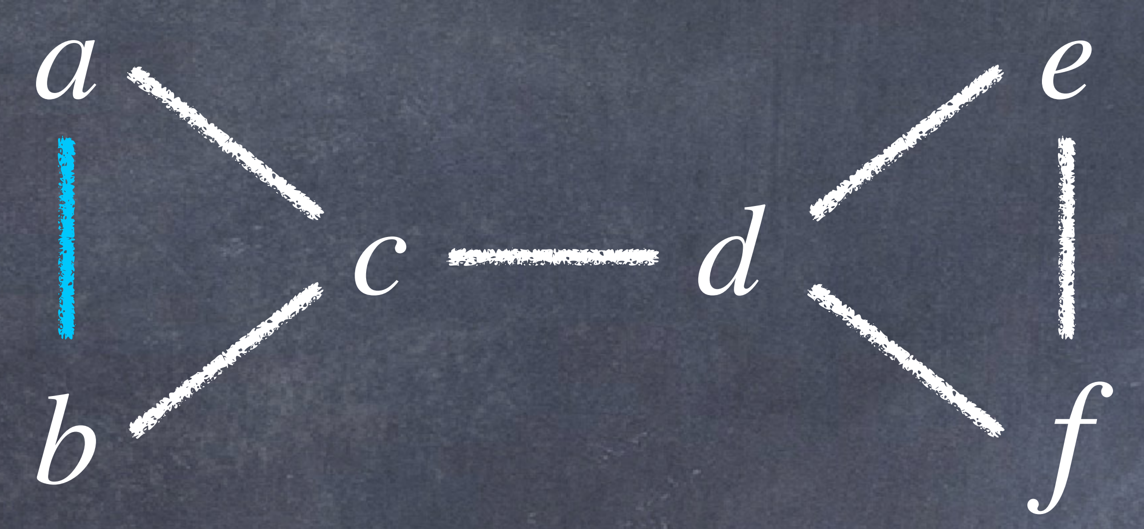

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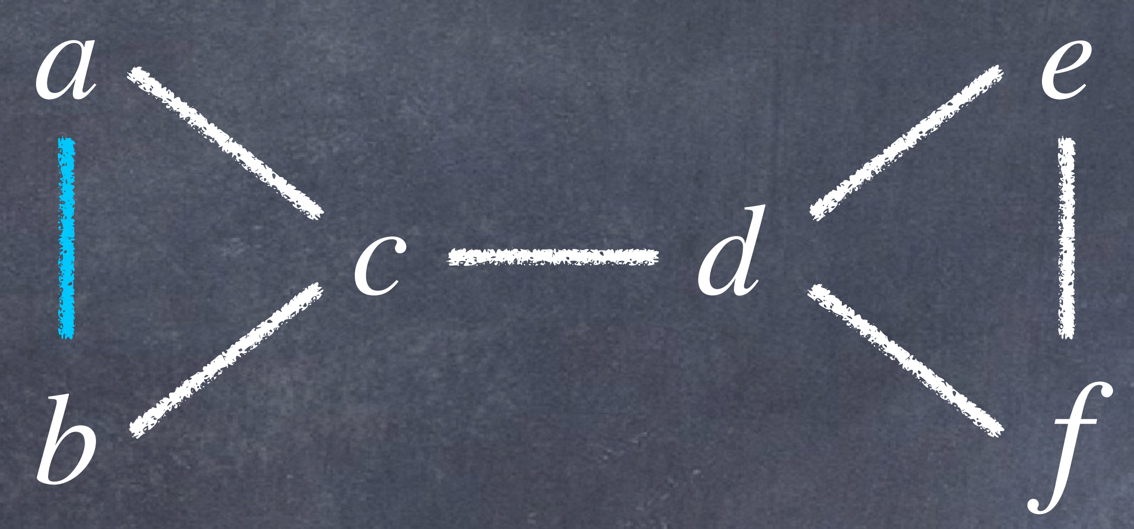

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$$abc \text{ ———— } def$$

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- Key observation: at anytime, the minimum degree $\geq k$

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$$\Pr[\text{not output a min-cut}] \leq \left(1 - \frac{2}{n(n-1)}\right)^{cn^2} \leq e^{-c/2}$$

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- Another definition: (double counting)

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- Warning: forget everything about conditional expectations

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- Do not assume any independence here

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$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X_i] = 1$$

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- $\mathbb{E}[T(n)] = \mathbb{E}\left[\sum_{1 \le i < j \le n} X_{ij}\right] = \sum_{1 \le i < j \le n} \mathbb{E}[X_{ij}]$

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- More general: concentration of measures...

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- $X_1, \ldots, X_N$ are independently identically distributed (i.i.d.)
- We may expect $\mathbb{E}[X_1 + X_2 + \cdots + X_N] = \mathbb{E}[N] \cdot \mathbb{E}[X_i]$
- Is it true?

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- How about randomly many independent r.v.s?

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- Clearly, at each round $\mathbb{E}[X_i] = 0$
- However, the gambler always stops with winning $\$1$ ($\sum X_i = 1$)

Enjoy the world of randomness !

Thank you

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- In particular, if $\mathbb{E}[X_k] = \mu$, $\mathbb{E}[X_1 + \cdots + X_N] = \mu \cdot \mathbb{E}[N]$

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- $\mathbb{E}[X_1 + \cdots + X_N] = \frac{1}{36} \cdot (21 + 21 \cdot 2 + \cdots + 21 \cdot 6) = \frac{21^2}{36} = \frac{49}{4}$

Random recurrence

KUW inequality

Quick-Select revisit

Coupon collector revisit

Yao's lemma

Monte Carlo vs. Las Vegas

Complexity class

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Thank you