[AI2613 Lecture 11] Brownian Motion, Brownian Bridge

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1 Brownian Motion

Brownian motion describes the random motion of small particles suspended in a liquid or in a gas. This process was named after the botanist Robert Brown, who observed and studied a jittery motion of pollen grains suspended in water under a microscope. Later, Albert Einstein gave a physical explanation of this phenomenon. In mathematics, Brownian motion is characterized by the *Wiener process*, named after Norbert Wiener, a famous mathematician and the originator of cybernetics.

To motivate the definition of Brownian motion, we start from the 1-D random walk starting from 0. Let Z_t be our position at time t and X_t be the move of the t-th step. The value of X_t is chosen from $\{-1, 1\}$ uniformly at random. Note that $Z_0 = 0$ and $Z_{t+1} = Z_t + X_t$. So $Z_T = \sum_{t=0}^{T-1} X_t$. Then we have

$$\mathbf{E}[Z_T] = 0 \text{ and } \mathbf{Var}[Z_T] = \sum_{t=0}^{T-1} \mathbf{Var}[X_t] = T.$$

Suppose now we move with every Δt seconds and with step length δ. Then our position at time T is $Z(T) = \delta \sum_{t=1}^{\frac{T}{\Delta t}} X_t$. We are interested in the behavior of the prcoess when $\Delta t \to 0$. We have

$$\mathbf{E}[Z(T)] = 0 \text{ and } \mathbf{Var}[Z(T)] = \delta^2 \sum_{t=1}^{\frac{T}{\Delta t}} \mathbf{Var}[X_t] = \delta^2 \cdot \frac{T}{\Delta t}.$$

We can identify the expectation and the variance of this process with the discrete random walk when $\Delta t \to 0$ by choosing $\delta = \sqrt{\Delta t}$. It follows from the central limit theorem that

$$Z(T) = \sqrt{\Delta t} \sum_{t=1}^{\frac{T}{\Delta t}} X_t \overset{\Delta t \to 0}{\longrightarrow} \sqrt{\Delta t}\mathcal{N}(0, \frac{T}{\Delta t}) = \mathcal{N}(0, T).$$

In other words, the "continuous" version of the 1-D random walk follows $\mathcal{N}(0, T)$ at time T. This is the basis of the Wiener process. Now we introduce its formal definition.

Definition 1 (Standard Brownian Motion / Wiener Process) *We say a stochastic process $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion or Wiener process if it satisfies*

- $W(0) = 0$;
- **Independent increments**: $\forall 0 \leq t_0 \leq t_1 \leq \cdots \leq t_n$, $W(t_1) - W(t_0)$, $W(t_2) - W(t_1), \ldots, W(t_n) - W(t_{n-1})$ *are mutually independent;*
- **Stationary increments**: $\forall s, t > 0$, $W(s + t) - W(s) \sim \mathcal{N}(0, t)$;

- *$W(t)$ is continuous almost surely.*[1]

Let $\{W(t)\}_{t\geq 0}$ be a a standard Brownian motion. If $\{X(t)\}_{t\geq 0}$ satisfies $X(t) = \mu \cdot t + \sigma W(t)$, we call $\{W(t)\}_{t\geq 0}$ a (μ, σ^2) Brownian motion.

Now we introduce another characterization of Brownian motion. First recall the notion of high dimensional Gaussian distribution. A vector of random variables $(X_1, X_2, \ldots, X_n)$ is said to be Gaussian iff $\forall a_1, a_2, \ldots, a_n$, $\sum_{i=1}^n a_i X_i$ is a one-dimensional Gaussian. Let $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$ where $\mu_i = \mathbf{E}[X_i]$. Let $\Sigma = \left(\mathrm{Cov}(X_i, X_j)\right)_{i,j}$. Then the probability density function f of $(X_1, X_2, \ldots, X_n)$ is

$$\text{for } x = (x_1, x_2, \ldots, x_n),\ f(x) = (2\pi)^{-\frac{n}{2}} \cdot |\det\Sigma|^{-\frac{1}{2}} \cdot e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1} (x-\mu)}.$$

To give another characterization of standard Brownian motion, we first define the *Gaussian process.*

Definition 2 (Gaussian Process) *A stochastic process $\{W(t)\}_{t\geq 0}$ is called Gaussian process if $\forall 0 \leq t_1 \leq t_2 \leq \cdots \leq t_n$, $(W(t_1), W(t_2), \ldots, W(t_n))$ is a Gaussian.*

Definition 3 (Standard Brownian Motion/Standard Wiener Process) *We say a stochastic process $\{W(t)\}_{t\geq 0}$ is a standard Brownian motion or Wiener process if it satisfies*

- *$\{W(t)\}_{t\geq 0}$ is an almost surely continuous Gaussian Process;*
- *$\forall s \geq 0$, $\mathbf{E}[W(s)] = 0$;*
- *$\forall 0 \leq s \leq t$, $\mathrm{Cov}(W(s), W(t)) = s$.*

Sometimes it is easier to use Definition 3 to show that a stochastic process is a Brownian motion. We now verify the equivalence between Definition 1 and Definition 3.

Proof. Given Definition 1, it is easy to know that $\mathbf{E}[W(s)] = 0$ for all $s \geq 0$ since $W(s) \sim \mathcal{N}(0, s)$. What we need is to verify that $\{W(t)\}_{t\geq 0}$ in Definition 1 is a Gaussian process and to compute the covariance of $W(s)$ and $W(t)$ in Definition 1.

Note that $\forall 0 \leq s < t$ and $\forall a, b$, we have

$$aW(s) + bW(t) = (a+b)W(s) + b(W(t) - W(s)).$$

Since $W(s)$ and $W(t) - W(s)$ are two independent Gaussian's, $aW(s) + bW(t)$ is still a Gaussian.

By the distributive law of covariance, for any $0 \leq s \leq t$, we have

$$\begin{aligned}\mathrm{Cov}(W(s), W(t)) &= \mathrm{Cov}\left(W(s), W(t) - W(s) + W(s)\right)\\ &= \mathrm{Cov}(W(s), W(t) - W(s)) + \mathrm{Cov}(W(s), W(s))\\ &= \mathbf{Var}[W(s)] = s.\end{aligned}$$

[1] Let Ω be the sample space. Then W can be viewed as a mapping from $\mathbb{R} \times \Omega$ to $\mathbb{R}$. The meaning of "$W(t)$ is continuous almost surely" is: $\exists \Omega_0 \subseteq \Omega$ with $\mathbf{Pr}[\Omega_0] = 1$ such that $\forall \omega \in \Omega_0$, $W(t, \omega)$ is continuous with regard to t.

Then we consider the counterpart. Given Definition 3, we can deduce the first and fourth property in Definition 1 directly. For any $0 \le t_{i-1} \le t_i \le t_{j-1} \le t_j$, we have

$$\begin{aligned} &\mathrm{Cov}(W(t_i) - W(t_{i-1}), W(t_j) - W(t_{j-1})) \\ =& \mathrm{Cov}(W(t_i), W(t_j)) + \mathrm{Cov}(W(t_{i-1}), W(t_{j-1})) \\ &- \mathrm{Cov}(W(t_i), W(t_{j-1})) - \mathrm{Cov}(W(t_{i-1}), W(t_j)) \\ =& t_i + t_{i-1} - t_i - t_{i-1} = 0, \end{aligned}$$

which yields the independence of $W(t_i) - W(t_{i-1})$ and $W(t_j) - W(t_{j-1})$. Thus, the $\{W(t)\}_{t\ge0}$ in Definition 3 satisfies independent increments.

It is easy to verify that $\forall s, t > 0$, $W(s+t) - W(s)$ is a Gaussian with mean 0. Note that

$$\begin{aligned} \mathbf{Var}\left[W(t+s) - W(s)\right] &= \mathbf{E}\left[(W(t+s) - W(s))^2\right] \\ &= \mathbf{E}\left[W(t+s)^2\right] + \mathbf{E}\left[W(s)^2\right] - 2\mathbf{E}\left[W(t+s)W(s)\right] \\ &= \mathbf{Var}\left[W(t+s)^2\right] + \mathbf{Var}\left[W(s)^2\right] - 2\mathrm{Cov}\left(W(t+s), W(s)\right) \\ &= t + s + s - 2s = t. \end{aligned}$$

Thus, the $\{W(t)\}_{t\ge0}$ in Definition 3 satisfies stationary increments. □

Example 1 *Suppose $\{W(t)\}_{t\ge0}$ is a standard Brownian motion. We claim that $\{X(t)\}_{t\ge0}$ is also a standard Brownian motion where $X(0) = 0$ and $X(t) = t \cdot W(\frac{1}{t})$ for $t > 0$.*

We verify the three requirements in Definition 3.

Since $X(t) = t \cdot W(\frac{1}{t})$ which is the compound of two (almost surely) continuous function, $\{X(t)\}_{t\ge0}$ is also continuous almost surely. For any $a_1, a_2, \dots, a_n$ and $t_1, t_2, \dots, t_n \ge 0$, $\sum_{i=1}^n a_i X(t_i) = \sum_{i=1}^n a_i t_i \cdot W(\frac{1}{t_i})$. Since $\{W(t)\}$ is standard Brownian motion, $\sum_{i=1}^n a_i t_i \cdot W(\frac{1}{t_i})$ is Gaussian. Thus, $\{X(t)\}_{t\ge0}$ is a Gaussian process. For $0 \le s < t$,

$$\begin{aligned} \mathrm{Cov}(X(s), X(t)) &= \mathrm{Cov}(sW(\frac{1}{s}), tW(\frac{1}{t})) \\ &= st \cdot \mathrm{Cov}(W(\frac{1}{s}), W(\frac{1}{t})) \\ &= st \cdot \frac{1}{t} = s. \end{aligned}$$

Thus, $\{X(t)\}_{t\ge0}$ is a standard Brownian motion.

Example 2 (Hitting time) *We consider the first hitting time of position b in a Brownian motion. Define $\tau_b \triangleq \inf\{t \ge 0 \mid W(t) > b\}$. For any $t > 0$,*

$$\begin{aligned} \mathbf{Pr}\left[\tau_b < t\right] &= \mathbf{Pr}\left[\tau_b < t \wedge W(t) > b\right] + \mathbf{Pr}\left[\tau_b < t \wedge W(t) < b\right] \\ &= \mathbf{Pr}\left[W(t) > b\right] + \mathbf{Pr}\left[W(t) < b \mid \tau_b < t\right] \cdot \mathbf{Pr}\left[\tau_b < t\right]. \end{aligned}$$

Note that $W(t) \sim \mathcal{N}(0, t)$. Let Φ be the cumulative distribution function of standard Gaussian distribution, that is, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^x e^{-\frac{t^2}{2}} dt$. Then

$$\mathbf{Pr}\left[W(t) > b\right] = \mathbf{Pr}\left[\frac{W(t)}{\sqrt{t}} > \frac{b}{\sqrt{t}}\right] = 1 - \Phi\left(\frac{b}{\sqrt{t}}\right).$$

Assume we have known the value of τ_b and $\tau_b < t$, we can regard $\{W(t)\}_{t \geq \tau_b}$ as a Brownian motion starting from b. Thus, as Figure 1 shows, $\mathbf{Pr}\left[W(t) < b \mid \tau_b < t\right] = \frac{1}{2}$.

By direct calculation, we have $\mathbf{Pr}\left[\tau_b < t\right] = 2\left(1 - \Phi\left(\frac{b}{\sqrt{t}}\right)\right)$.

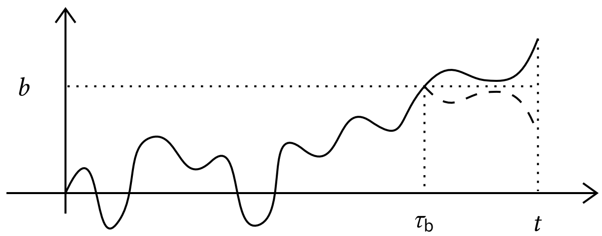

Figure 1: A hitting time and the reflection principle

2 Brownian Bridge

Consider a Brownian motion starting from $W(0) = 0$ and ending at $W(u) = x$. Conditioned on fixed $W(0)$ and $W(u)$, what is the distribution of $W(t)$? By definition, for $t < u$, conditioned on $W(u) = x$, $W(t)$ is a Gaussian. So it is sufficient to compute its mean and variance. As Figure 2 shows, a natural conjecture is that for $0 \leq t \leq u$, $\mathbf{E}\left[W(t) \mid W(u)\right] = \frac{t}{u}W(u)$.

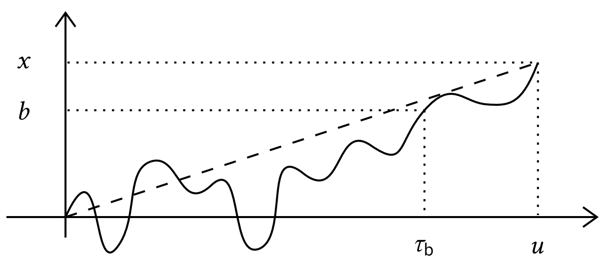

Figure 2: A Brownian bridge

To verify this, we first prove the following proposition.

Proposition 4 *For any $0 \leq t \leq u$, $W(t) - \frac{t}{u}W(u)$ is independent of $W(u)$.*

Proof.

$$\begin{aligned}\mathrm{Cov}(W(t) - \frac{t}{u}W(u), W(u)) &= \mathrm{Cov}(W(t), W(u)) - \frac{t}{u}\mathbf{Var}\left[W(u)\right] \\ &= t - \frac{t}{u} \cdot u = 0.\end{aligned}$$

The proposition follows from the fact that the two Gaussians are independent iff their covariance is zero. □

Thus, we have

$$0 = \mathbf{E}\left[W(t) - \frac{t}{u}W(u)\right] = \mathbf{E}\left[W(t) - \frac{t}{u}W(u) \;\middle|\; W(u)\right] = \mathbf{E}\left[W(t) \mid W(u)\right] - \frac{t}{u}W(u).$$

This confirms the conjecture that for $0 \le t \le u$, $\mathbf{E}[W(t) \mid W(u)] = \frac{t}{u}W(u)$. Then we consider the variance of $W(t)$ conditioned on $W(u)$.

$$\begin{aligned}
\mathbf{Var}[W(t) \mid W(u)] &= \mathbf{E}\left[(W(t) - \mathbf{E}[W(t) \mid W(u)])^2 \,\middle|\, W(u)\right] \\
&= \mathbf{E}\left[\left(W(t) - \frac{t}{u}W(u)\right)^2 \,\middle|\, W(u)\right] \\
&= \mathbf{E}\left[\left(W(t) - \frac{t}{u}W(u)\right)^2\right] \\
&= \mathbf{E}\left[W(t)^2\right] + \frac{t^2}{u^2}\mathbf{E}\left[W(u)^2\right] - 2\frac{t}{u}\mathbf{E}[W(t)W(u)] \\
&= \frac{t(u-t)}{u}.
\end{aligned}$$

Finally, to characterize the distribution of $\{W(t)\}$ conditioned on $W(u) = x$, we compute the covariance. Let $p_{W(t)}$ be the probability density function of $W(t)$. For any $s \le t$,

$$\begin{aligned}
\mathrm{Cov}(W(s), W(t)|W(u)) &= \mathbf{E}[W(s) \cdot W(t) \mid W(u)] - \mathbf{E}[W(s) \mid W(u)] \cdot \mathbf{E}[W(t) \mid W(u)] \\
&= \int_{\mathbb{R}} y \cdot \mathbf{E}[W(s) \mid W(t) = y, W(u)] \cdot p_{W(t)}(y|W(u))dy - \frac{st}{u^2}W(u)^2 \\
&= \int_{\mathbb{R}} y \cdot \frac{s}{t}y \cdot p_{W(t)}(y|W(u))dy - \frac{st}{u^2}W(u)^2 \\
&= \frac{s}{t}\mathbf{E}\left[W(t)^2|W(u)\right] - \frac{st}{u^2}W(u)^2 \\
&= \frac{s(u-t)}{u}.
\end{aligned}$$

To sum up, conditioned on $W(u)$, $\{W(t)\}$ has the following three properties:

- For $t \in [0, u]$, $\mathbf{E}[W(t) \mid W(u)] = \frac{t}{u}W(u)$.
- For $t \in [0, u]$, $\mathbf{Var}[W(t) \mid W(u)] = \frac{t(u-t)}{u}$.
- For any $0 \le s \le t \le u$, $\mathrm{Cov}(W(s), W(t)|W(u)) = \frac{s(u-t)}{u}$.

We call the Brownian motion $\{W(t)\}_{t \ge 0}$ which ends at $W(u) = x$ a Brownian bridge. Furthermore, we define the standard Brownian bridge.

Definition 5 (Standard Brownian Bridge) *A standard Brownian motion ending at $W(1) = 0$ is called a standard Brownian bridge.*

We can verify that letting $X(t) = W(t) - tW(1)$ where $\{W(t)\}$ is a standard Brownian motion, then $\{X(t)\}$ is a standard Brownian Bridge.

Example 3 (Hitting Time in a Brownian Bridge) *Let $\{W(t)\}_{t \ge 0}$ be a standard Brownian motion. Let $\tau_b \triangleq \inf\{t \ge 0 \mid W(t) > b\}$. Then we compute $\mathbf{Pr}[\tau_b < u \mid W(u) = x]$. Note that if $b < x$, $\mathbf{Pr}[\tau_b < u \mid W(u) = x] = 1$. Let ψ be the probability density function of standard Gaussian distribution, that is,*

$\psi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}$. *If* $b > x$, *letting* $\mathrm{d}x = [x, x+h]$ *where* h *is infinitesimal, we have*

$$\begin{aligned}\mathbf{Pr}\left[\tau_b < u \mid W(u) = x\right] &= \frac{\mathbf{Pr}\left[\tau_b < u \wedge W(u) \in \mathrm{d}x\right]}{\mathbf{Pr}\left[W(u) \in \mathrm{d}x\right]} \\ &= \frac{\mathbf{Pr}\left[\tau_b < u\right] \cdot \mathbf{Pr}\left[W(u) \in \mathrm{d}x \mid \tau_b < u\right]}{\frac{1}{\sqrt{u}}\psi\left(\frac{x}{\sqrt{u}}\right)\mathrm{d}x}.\end{aligned}$$

If we have known the value of τ_b *and* $\tau_b < u$, *we can regard* $\{W(u)\}_{t \geq \tau_b}$ *as a Brownian motion starting from* b. *Then we have*

$$\begin{aligned}\mathbf{Pr}\left[\tau_b < u\right] \cdot \mathbf{Pr}\left[W(u) \in \mathrm{d}x \mid \tau_b < u\right] &= \mathbf{Pr}\left[\tau_b < u\right] \cdot \mathbf{Pr}\left[W(u) \in 2b - \mathrm{d}x \mid \tau_b < u\right] \\ &= \mathbf{Pr}\left[\tau_b < u \wedge W(u) \in 2b - \mathrm{d}x\right] \\ &= \mathbf{Pr}\left[W(u) \in 2b - \mathrm{d}x\right] \\ &= \frac{1}{\sqrt{u}}\psi\left(\frac{2b-x}{\sqrt{u}}\right)\mathrm{d}x\end{aligned}$$

Thus, when $b > x$, $\mathbf{Pr}\left[\tau_b < u \mid W(u) = x\right] = \frac{\psi\left(\frac{2b-x}{\sqrt{u}}\right)}{\psi\left(\frac{x}{\sqrt{u}}\right)} = e^{-\frac{2b(b-x)}{u}}$.

When $b = x$, *we have*

$$\mathbf{Pr}\left[\tau_b < u \mid W(u) = b\right] = \frac{\mathbf{Pr}\left[\tau_b < u \wedge W(u) \in \mathrm{d}b\right]}{\mathbf{Pr}\left[W(u) \in \mathrm{d}b\right]}.$$

Note that

$$\mathbf{Pr}\left[\tau_b < u \wedge W(u) \in \mathrm{d}b\right] = \mathbf{Pr}\left[\tau_b < u\right] - \mathbf{Pr}\left[\tau_b < u \wedge W(u) > b + h\right] - \mathbf{Pr}\left[\tau_b < u \wedge W(u) < b\right]. \tag{1}$$

By Example 2, we have $\mathbf{Pr}\left[\tau_b < u\right] = 2\left(1 - \Phi\left(\frac{b}{\sqrt{u}}\right)\right)$. *Note that*

$$\begin{aligned}\mathbf{Pr}\left[\tau_b < u \wedge W(u) > b + h\right] &= \mathbf{Pr}\left[W(u) > b + h\right] \\ &= 1 - \Phi\left(\frac{b}{\sqrt{u}}\right) - \mathbf{Pr}\left[W(u) \in \mathrm{d}b\right].\end{aligned}$$

And

$$\begin{aligned}\mathbf{Pr}\left[\tau_b < u \wedge W(u) < b\right] &= \mathbf{Pr}\left[\tau_b < u\right] \cdot \mathbf{Pr}\left[W(u) < b \mid \tau_b < u\right] \\ &= \frac{1}{2}\mathbf{Pr}\left[\tau_b < u\right] = 1 - \Phi\left(\frac{b}{\sqrt{u}}\right).\end{aligned}$$

Thus, Equation (1) equals to $\mathbf{Pr}\left[W(u) \in \mathrm{d}b\right]$ *and* $\mathbf{Pr}\left[\tau_b < u \mid W(u) = b\right] = 1$.