[AI2613 Lecture 6] Galton-Watson Process, 1-D Random Walk, 2-SAT

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1 Galton-Watson Process

The model was formulated by F. Galton in the study of the survival and extinction of family names. In the nineteenth century, there was concern amongst the Victorians that aristocratic surnames were becoming extinct. In 1873, Galton originally posed the question regarding the probability of such an event, and later H. W. Watson replied with a solution.

Using more modern terms, the process can be defined formally as follows:

Definition 1 (Galton-Watson Process) *Suppose that all the individuals reproduce independently of each other and have the same offspring distribution. More precisely, let G_t denote the number of individuals of t-th generation:*

- *We start from the zero generation. For convenience, let $G_0 = 1$.*
- *Each individual of generation t gives birth to a random number of children of generation $t+1$. That is, $\forall t \geq 0$ and $i \in [G_t]$, let $X_{t,i}$ denote the number of children of the i-th individual in the t-th generation. Then $\{X_{t,i}\}$ is an array of i.i.d. random variables with $\mathbf{Pr}\left[X_{t,i} = k\right] = p_k$.*
- *All individuals of generation $t+1$ are children of individuals of generation t:*

$$G_{t+1} = \sum_{i=1}^{G_t} X_{t,i}$$

It is clear that the process $\{G_t\}_{t\geq 0}$ is a Markov chain.

Denote by ρ the probability of extinction, namely

$$\rho \triangleq \mathbf{Pr}\left[\text{extinction}\right] = \mathbf{Pr}\left[\cup_{t\geq 1}\{G_t = 0\}\right].$$

Then the question is to determine the value of ρ. First we consider two trivial situations:

- When $p_0 = 0$, it is clear that there will be offspring and $\rho = 0$.
- When $p_0 > 0$ and $p_0 + p_1 = 1$, we can verify that $\rho = 1$. We know that

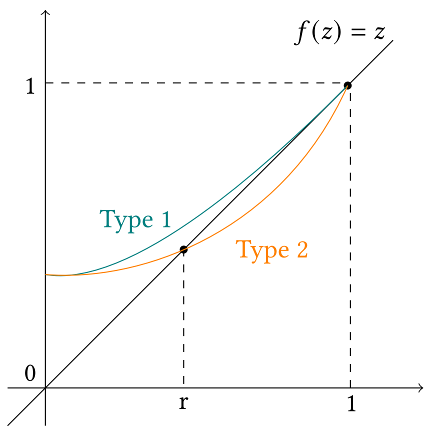
  $$\mathbf{E}\left[G_{t+1} | G_t\right] = p_1 \cdot G_t.$$

  Compute the expectation of both sides, we have $\mathbf{E}\left[G_{t+1}\right] = p_1 \mathbf{E}\left[G_t\right]$. This yields that when $t \to \infty$, $\mathbf{Pr}\left[G_t \geq 1\right] \leq \mathbf{E}\left[G_t\right] = p_1^t \mathbf{E}\left[G_0\right] \to 0$.

Then we assume that $p_0 > 0$ and $p_0 + p_1 < 1$. By the independence of each individual and the Markov property, we can calculate ρ as follows:

$$\rho = \sum_{k=0}^{\infty} \mathbf{Pr}\left[\text{extinction} \wedge G_1 = k\right] = \sum_{k=0}^{\infty} \mathbf{Pr}\left[\text{extinction} | G_1 = k\right] p_k = \sum_{k=0}^{\infty} \rho^k p_k. \quad (1)$$

Let $\psi(z) \triangleq \sum_{k=0}^{\infty} p_k z^k$. Then Equation (1) yields that ρ is a fixed point of ψ, i.e., $\psi(\rho) = \rho$. By direct calculation we know ψ is an increasing and convex function on $[0, 1]$ with $\psi(0) = p_0$ and $\psi(1) = 1$. Then there can be two types of ψ depending on whether $\psi'(1)$ is larger than 1 as the following figure shows.



When $\psi'(1) = \sum_{k=1}^{\infty} k p_k = \mathbf{E}\left[X_{t-i}\right] \le 1$, $z = 1$ is the only fixed point of ψ which corresponds to the Type 1 in the figure. That is to say, when $\mathbf{E}\left[X_{t-i}\right] \le 1$, we have $\rho = 1$.

When $\mathbf{E}\left[X_{t-i}\right] > 1$ (Type 2), although there are two fixed points of ψ: r and 1, we claim that $\rho = r$ rather than 1 by showing that $\rho \le r$. Let $q_t \triangleq \mathbf{Pr}\left[G_t = 0\right]$. Then $q_t \le q_{t+1} < 1$ since $G_t = 0$ can always yields $G_{t+1} = 0$. We induct on t to show that $q_t \le r$:

- When $t = 0$, it is obvious that $q_0 = 0 < r$.
- Assume that $q_t \le r$. Since $q_{t+1} = \sum_{k=0}^{\infty} p_k q_t^k = \psi(q_t)$ and ψ is an increasing function, $q_{t+1} = \psi(q_t) \le \psi(r) = r$.

We know that $\rho = \lim_{t \to \infty} q_t$ and $q_t \le r$ for all $t \ge 0$. Thus $\rho \le r$. However, we have shown that ρ is a fixed point of ψ. So $\rho = r$ when $\mathbf{E}\left[X_{t-i}\right] > 1$. In conclusion, $\rho = 1$ iff $\mathbf{E}\left[X_{t-i}\right] \le 1$.

2 1-D Random Walk

Consider the following one-dimensional random walk:

Let X_t be the position at time step t. Let $T_{i \to j}$ be the first hitting time of state j starting from i, that is, $T_{i\to j} = \min\{t > 0 | X_t = j \land X_0 = i\}$. Define event $\mathscr{A} = $ [the first step is to the right]. Then we consider the problem that when will this Markov chain be recurrent. Note that

$$
\begin{aligned}
\mathbf{Pr}\left[T_{0\to 0} < \infty\right] &= \mathbf{Pr}\left[T_{0\to 0} < \infty | \bar{\mathscr{A}}\right] \mathbf{Pr}\left[\bar{\mathscr{A}}\right] + \mathbf{Pr}\left[T_{0\to 0} < \infty | \mathscr{A}\right] \mathbf{Pr}\left[\mathscr{A}\right] \\
&= (1-p)\cdot 1 + p \cdot \mathbf{Pr}\left[T_{1\to 0} < \infty\right], \qquad (2)\\
\mathbf{Pr}\left[T_{1\to 0} < \infty\right] &= \mathbf{Pr}\left[T_{1\to 0} < \infty | \bar{\mathscr{A}}\right] \mathbf{Pr}\left[\bar{\mathscr{A}}\right] + \mathbf{Pr}\left[T_{1\to 0} < \infty | \mathscr{A}\right] \mathbf{Pr}\left[\mathscr{A}\right] \\
&= (1-p)\cdot 1 + p \cdot \mathbf{Pr}\left[T_{2\to 0} < \infty\right], \qquad (3)\\
\mathbf{Pr}\left[T_{2\to 0} < \infty\right] &= \mathbf{Pr}\left[T_{2\to 1} < \infty \land T_{1\to 0} < \infty\right] \\
&= \mathbf{Pr}\left[T_{2\to 1} < \infty\right] \cdot \mathbf{Pr}\left[T_{1\to 0} < \infty\right] \\
&= \mathbf{Pr}\left[T_{1\to 0} < \infty\right]^2. \qquad (4)
\end{aligned}
$$

Let $y \triangleq \mathbf{Pr}\left[T_{1\to 0} < \infty\right]$ for brevity. Combine Equation (3) and Equation (4), we have $y = 1 - p + py^2$ which then yields $y = 1$ or $y = \frac{1-p}{p}$. By Equation (2), $\mathbf{Pr}\left[T_{0\to 0} < \infty\right] = 1$ or $2 - 2p$.

- When $p < \frac{1}{2}$, $2 - 2p$ is meaningless as a probability. So $\mathbf{Pr}\left[T_{0\to 0} < \infty\right] = 1$ and the Markov chain is recurrent.
- When $p = \frac{1}{2}$, $2 - 2p = 1$. The Markov chain is also recurrent in this situation.
- When $p > \frac{1}{2}$, we verify that $\mathbf{Pr}\left[T_{0\to 0} < \infty\right] < 1$, and therefore $\mathbf{Pr}\left[T_{0\to 0} < \infty\right] = 2 - 2p$. Let $\{\Delta_k\}_{k=0}^{\infty}$ be a sequence of i.i.d. random variables with

$$
\Delta_k = \begin{cases} +1, & \text{w.p. } p \\ -1, & \text{w.p. } 1-p \end{cases}.
$$

  Given a sufficiently large $n \in \mathbb{N}$, we can walk to n from 0 in n steps (i.e. $X_n = n$) with probability $p^n > 0$. Assume that we have arrived at n, consider the probability that we go back to 0 from n in exactly k steps. Apparently, this probability is zero when $k < n$. For every $k \geq n$, we upper bound the probability $\mathbf{Pr}\left[T_{n\to 0}=k\right]$:

$$
\begin{aligned}
\mathbf{Pr}\left[T_{n\to 0}=k\right] &\le \mathbf{Pr}\left[\sum_{t=1}^{k}\Delta_t=-n\right]\\
&\le \mathbf{Pr}\left[\sum_{t=1}^{k}\Delta_t-\mathbf{E}\left[\sum_{t=1}^{k}\Delta_t\right]\le -n-\mathbf{E}\left[\sum_{t=1}^{k}\Delta_t\right]\right]\\
&\le \exp\left\{-\frac{2k\left(\frac{n+(2p-1)k}{k}\right)^2}{4}\right\}.
\end{aligned}
$$

where the third inequality follows from the Hoeffding's inequality.

Then we calculate the probability that we can go back from n to 0. By union bound,

$$
\begin{aligned}
\mathbf{Pr}\left[T_{n\to 0}<\infty\right] &= \mathbf{Pr}\left[\bigcup_{k\ge n}\left[T_{n\to 0}=k\right]\right]\\
&\le \sum_{k=n}^{\infty}\mathbf{Pr}\left[T_{n\to 0}=k\right]\\
&\le \exp\{-(2p-1)n\}\sum_{k=n}^{\infty}\exp\left\{-\frac{n^2}{2k}-\frac{(2p-1)^2k}{2}\right\}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\sum_{k=n}^{\infty}\exp\left\{-\frac{n^2}{2k}\right\}\cdot\exp\left\{-\frac{(2p-1)^2k}{2}\right\} &\le \sum_{k=n}^{\infty}\exp\left\{-\frac{(2p-1)^2k}{2}\right\}\\
&= \frac{\exp\left\{-\frac{(2p-1)^2}{2}n\right\}}{1-\exp\left\{-\frac{(2p-1)^2}{2}\right\}}
\end{aligned}
$$

Thus,

$$
\mathbf{Pr}\left[T_{n\to 0}<\infty\right]\le \frac{\exp\left\{-\frac{(2p-1)^2}{2}n-(2p-1)n\right\}}{1-\exp\left\{-\frac{(2p-1)^2}{2}\right\}}. \tag{5}
$$

We can find a sufficiently large constant n such that $\mathbf{Pr}\left[T_{n\to 0}<\infty\right]<1$ since the RHS of Equation (5) is exponentially small with regard to n. So for sufficiently large n, the probability that we walk to n and never come back to 0 is larger than $p^n\cdot\mathbf{Pr}\left[T_{n\to 0}=\infty\right]>0$. Thus, this Markov chain is transient.

Now we verify that the Markov chain is positive recurrent when $p<\frac{1}{2}$ and null recurrent when $p=\frac{1}{2}$. Note that

$$
T_{0\to 0} = \mathbb{1}[\bar{\mathscr{A}}]\cdot 1+\mathbb{1}[\mathscr{A}](1+T_{1\to 0}) \tag{6}
$$

$$
T_{1\to 0} = \mathbb{1}[\bar{\mathscr{A}}]\cdot 1+\mathbb{1}[\mathscr{A}](1+T_{2\to 0}) \tag{7}
$$

$$
T_{2\to 0} = T_{2\to 1}+T_{1\to 0} = 2T_{1\to 0}. \tag{8}
$$

Taking the expectation of Equation (7) and combining with Equation (8), we have

$$\mathbf{E}\left[T_{1\to 0}\right] = 1 - p + p(1 + 2\mathbf{E}\left[T_{1\to 0}\right]),$$

which yields $\mathbf{E}\left[T_{1\to 0}\right] = \frac{1}{1-2p}$. Take the expectation of Equation (6), we get $\mathbf{E}\left[T_{0\to 0}\right] = \frac{1-p}{1-2p}$. Thus:

- When $p = \frac{1}{2}$, $\mathbf{E}\left[T_{0\to 0}\right] = \infty$ and the Markov chain is null recurrent.
- When $p < \frac{1}{2}$, $\mathbf{E}\left[T_{0\to 0}\right] < \infty$ and the Markov chain is positive recurrent.

3 2-SAT

SAT is the problem of determining whether a CNF formula has satisfying assignments. k-SAT is the special cases of SAT that the clauses of the CNF formula consist of exact k literals. For example,

$$\phi = (x \vee y) \wedge (y \vee \bar{z}) \wedge (\bar{x} \vee z)$$

is a 2-CNF formula and $x = y = z = 1$ is one of its satisfying assignments. SAT is **NP**-complete and we have k-SAT $\in$ **NP** for $k \geq 3$. One can use an algorithm for finding strongly connected components to solve 2- SAT problem in linear time. Nevertheless, we introduce a simple randomized algorithm that can also solve this problem in polynomial-time with high probability.

We will extend the algorithm to solving 3-SAT in the homework!

Let ϕ be a 2-CNF formula and $V = \{v_1, v_2, ..., v_n\}$ be its set of variables The algorithm runs as follows:

- Pick an arbitrary assignment $\sigma_0 : V \longrightarrow \{\text{true,false}\}$.
- For $t = 0, 1, 2, \ldots, 100n^2$:

  If σ_t satisfies ϕ, output σ_t;

  Else, pick an arbitrary unsatisfying clause, say $c = x \vee y$. Choose from $\{x, y\}$ uniformly at random and flip the assignment of the chosen literal. Let σ_{t+1} be the flipped assignment.
- Output "ϕ is not satisfiable".

Claim 2 *This algorithm outputs the correct answer with probability at least* $1 - \frac{1}{100}$.

Proof. It is clear that if a 2-SAT instance has no solution then our algorithm will always give the correct answer. So we consider the probability that our algorithm outputs no solution conditioned on that the instance indeed has a satisfying assignment.

Our algorithm produces $100n^2 + 1$ assignments $\sigma_0, \sigma_1, \ldots, \sigma_{100n^2}$. We claim that with probability at least $1 - \frac{1}{100}$, some of σ_k for $k \in \left\{0, \ldots, 100n^2 + 1\right\}$

is a satisfying assignment. The argument here, at first glance, is a bit weird. We fix an *arbitrary* $\sigma : V \to \{\texttt{true}, \texttt{false}\}$ satisfying assignment. We in fact prove the following: For large enough k, conditioned on the event that none of $\sigma_0, \sigma_1, \dots, \sigma_k$ is a satisfying assignment, $\sigma_{k+1} = \sigma$ holds with high probability.

Let $\{X_t\}_{t=0}^{100n^2}$ be a random variable sequence that

$$X_t \triangleq |\{v \in V : \sigma_t(v) = \sigma(v)\}|.$$

Note that $\{X_t\}_{t=0}^{100n^2}$ is not a Markov chain since it only contains partial information of σ_t and we cannot determine the distribution of X_{t+1} given X_t.

First we verify that $\mathbf{Pr}[X_{t+1} = X_t + 1 \mid \sigma_t] \ge \frac{1}{2}$ [1] and $\mathbf{Pr}[X_{t+1} = X_t - 1 \mid \sigma_t] \le \frac{1}{2}$. WLOG assume we chose the clause $c = x \vee y$ in round t. Since c is not satisfied by σ_t, we have $\sigma_t(x) = \sigma_t(y) = \texttt{false}$. Similarly, $x \vee y$ is satisfying under σ, so there are three possible assignments of $\sigma(x)$ and $\sigma(y)$:

[1] Let Y be a random variable. Then function $\mathbf{Pr}[\cdot \mid Y] : \mathrm{Ran}(Y) \to \mathbb{R}$ is defined by $\mathbf{Pr}[\cdot \mid Y] = \mathbf{E}[\mathbb{1}[\cdot] \mid Y]$. Note that $\mathbf{Pr}[\cdot \mid Y]$ is a random variable. Here we slightly abuse the notations and denote the event "$\forall a \in \mathrm{Ran}(Y), \mathbf{Pr}[\cdot \mid Y = a] \ge \frac{1}{2}$" as $\mathbf{Pr}[\cdot \mid Y] \ge \frac{1}{2}$.

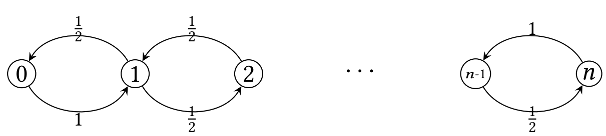
- If $\sigma(x) = \texttt{true}$ and $\sigma(y) = \texttt{false}$, $\mathbf{Pr}[X_{t+1} = X_t + 1 \mid \sigma_t] = \mathbf{Pr}[\text{flip } x] = \frac{1}{2}$ and $\mathbf{Pr}[X_{t+1} = X_t - 1 \mid \sigma_t] = \mathbf{Pr}[\text{flip } y] = \frac{1}{2}$.
- If $\sigma(x) = \texttt{false}$ and $\sigma(y) = \texttt{true}$, we have $\mathbf{Pr}[X_{t+1} = X_t + 1 \mid \sigma_t] = \mathbf{Pr}[X_{t+1} = X_t - 1 \mid \sigma_t] = \frac{1}{2}$ similarly.
- If $\sigma(x) = \texttt{true}$ and $\sigma(y) = \texttt{true}$, $\mathbf{Pr}[X_{t+1} = X_t + 1 \mid \sigma_t] = \mathbf{Pr}[\text{flip } x \text{ or } y] = 1$.

Thus we have $\mathbf{Pr}[X_{t+1} = X_t + 1 \mid \sigma_t] \ge \frac{1}{2}$ on condition that none of $\sigma_0, \sigma_1, \dots, \sigma_t$ is a satisfying assignment.

Consider the 1-D random walk $\{Y_t\}_{t\ge 0}$ on $[n] \cup \{0\}$ that $Y_0 = X_0$ and for $Y_t \notin \{0, 1\}$

$$Y_{t+1} = \begin{cases} Y_t + 1, & \text{w.p. } \frac{1}{2} \\ Y_t - 1, & \text{w.p. } \frac{1}{2} \end{cases}.$$

If $Y_t = 0$, $Y_{t+1} = Y_t + 1$ w.p. 1 and if $Y_t = n$, then $Y_{t+1} = Y_t - 1$ w.p. 1.



Then we have[2]

$$\begin{aligned}\mathbf{Pr}[\text{the algorithm is correct}] &\ge \mathbf{Pr}\left[\exists t \in [0, 100n^2]\ s.t. X_t = n\right] \\ &\ge \mathbf{Pr}\left[\exists t \in [0, 100n^2]\ s.t. Y_t = n\right]. \qquad (9)\end{aligned}$$

[2] The second inequality can be verified by constructing a coupling which satisfies $Y_t \ge X_t$ for all $t \ge 0$. The existence of such coupling is guaranteed by $\mathbf{Pr}[X_{t+1} = X_t + 1 \mid \sigma_t] \ge \mathbf{Pr}[Y_{t+1} = Y_t + 1]$. Specifically, if there is one false and one true in $\{\sigma(x), \sigma(y)\}$, then Y_{t+1} moves the same as X_{t+1}. If $\sigma(x) = \sigma(y)$ =true, then Y_{t+1} moves +1 or −1 uniformly at random.

Assume that $Y_0 = X_0 = i$. Let $T_{i\to n}$ be the first hitting time of n from i. Then

$$T_{i\to n} = \sum_{k=i}^{n-1} T_{k\to k+1}.$$

For $i > 0$, we have

Recall $\mathscr{A}$ = [the first step is to the right].

$$\begin{aligned} T_{i\to i+1} &= \mathbb{1}[\mathscr{A}] + \mathbb{1}[\bar{\mathscr{A}}](1 + T_{i-1\to i+1}) \\ &= \mathbb{1}[\mathscr{A}] + \mathbb{1}[\bar{\mathscr{A}}](1 + T_{i-1\to i} + T_{i\to i+1}) \end{aligned}$$

Taking the expectation of both sides, we have $\mathbf{E}[T_{i\to i+1}] = 2 + \mathbf{E}[T_{i-1\to i}]$.
Note that $T_{0\to 1} = 1$, then

$$\mathbf{E}[T_{i\to n}] = \sum_{k=i}^{n-1} \mathbf{E}[T_{k\to k+1}] = \sum_{k=i}^{n-1} 2k + 1 = n^2 - i^2 \leq n^2.$$

Then we apply the Markov's inequality to give a lower bound for $\mathbf{Pr}\left[\exists t \in [0, 100n^2]\ s.t. Y_t = n\right]$:

$$\begin{aligned} 1 - \mathbf{Pr}\left[\exists t \in [0, 100n^2]\ s.t. Y_t = n\right] &= \mathbf{Pr}\left[T_{Y_0\to n} > 100n^2\right] \\ &\leq \frac{\mathbf{E}\left[T_{Y_0\to n}\right]}{100n^2} \leq \frac{1}{100}. \end{aligned}$$

By Equation (9), we know that $\mathbf{Pr}$ [the algorithm is correct] is lower bounded by $1 - \frac{1}{100}$. □