[AI2613 Lecture 11] Brownian Motion

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1 Brownian Motion

Brownian motion describes the random motion of small particles suspended in a liquid or in a gas. This process was named after the botanist Robert Brown, who observed and studied a jittery motion of pollen grains suspended in water under a microscope. Later, Albert Einstein gave a physical explanation of this phenomenon. In mathematics, Brownian motion is characterized by the *Wiener process*, named after Norbert Wiener, a famous mathematician and the originator of cybernetics.

To motivate the definition of Brownian motion, we start from the 1-D random walk starting from 0. Let Z_t be our position at time t and X_t be the move of the t-th step. The value of X_t is chosen from $\{-1, 1\}$ uniformly at random. Note that $Z_0 = 0$ and $Z_{t+1} = Z_t + X_t$. So $Z_T = \sum_{t=0}^{T-1} X_t$. Then we have

$$\mathbf{E}[Z_T] = 0 \text{ and } \mathbf{Var}[Z_T] = \sum_{t=0}^{T-1} \mathbf{Var}[X_t] = T.$$

Suppose now we move with every Δt seconds and with step length δ. Then our position at time T is $Z(T) = \delta \sum_{t=1}^{\frac{T}{\Delta t}} X_t$. We are interested in the behavior of the prcoess when $\Delta t \to 0$. We have

$$\mathbf{E}[Z(T)] = 0 \text{ and } \mathbf{Var}[Z(T)] = \delta^2 \sum_{t=1}^{\frac{T}{\Delta t}} \mathbf{Var}[X_t] = \delta^2 \cdot \frac{T}{\Delta t}.$$

We can identify the expectation and the variance of this process with the discrete random walk when $\Delta t \to 0$ by choosing $\delta = \sqrt{\Delta t}$. It follows from the central limit theorem that

$$Z(T) = \sqrt{\Delta t} \sum_{t=1}^{\frac{T}{\Delta t}} X_t \overset{\Delta t \to 0}{\longrightarrow} \sqrt{\Delta t} \mathcal{N}(0, \frac{T}{\Delta t}) = \mathcal{N}(0, T).$$

In other words, the "continuous" version of the 1-D random walk follows $\mathcal{N}(0, T)$ at time T. This is the basis of the Wiener process. Now we introduce its formal definition.

Definition 1 (Standard Brownian Motion / Wiener Process)**.** *We say a stochastic process $\{W(t)\}_{t \geq 0}$ is a standard Brownian motion or Wiener process if it satisfies*

- $W(0) = 0$;

- **Independent increments**: $\forall 0 \le t_0 \le t_1 \le \cdots \le t_n$, $W(t_1) - W(t_0)$, $W(t_2) - W(t_1)$, $\dots$, $W(t_n) - W(t_{n-1})$ *are mutually independent;*
- **Stationary increments**: $\forall s, t > 0$, $W(s+t) - W(s) \sim \mathcal{N}(0, t)$;
- $W(t)$ *is continuous almost surely.*[1]

[1] Let Ω be the sample space. Then W can be viewed as a mapping from $\mathbb{R} \times \Omega$ to $\mathbb{R}$. The meaning of "$W(t)$ is continuous almost surely" is: $\exists \Omega_0 \subseteq \Omega$ with $\mathbf{Pr}[\Omega_0] = 1$ such that $\forall \omega \in \Omega_0$, $W(t, \omega)$ is continuous with regard to t.

Recall that the probability density of the Gaussian distribution $N(\mu, \sigma^2)$ is

$$f_{N(\mu,\sigma^2)}(x) = \frac{1}{\sigma\sqrt{2\pi}} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right).$$

We use $\Phi(\cdot)$ to denote the CDF of $N(0,1)$, namely $\Phi(t) = \int_{-\infty}^{t} f_{N(0,1)}(x)\,\mathrm{d}x$.

In the following, we use $f_t(x)$ to denote the probability density of $N(0,t)$. For any $t_1 \le t_2 \le \dots t_n$, the joint density of $W(t_1), W(t_2), \dots, W(t_n)$ is

$$f(x_1, \dots, x_n) = f_{t_1}(x_1) f_{t_2 - t_1}(x_2 - x_1) \dots f_{t_n - t_{n-1}}(x_n - x_{n-1})$$

Example 1. *Let $0 \le s \le t$. We can compute the conditional distribution of $X(s)$ when $X(t) = y$. We use $f_{s|t}(x|y)$ to denote the probability density of $X(s) = x$ conditioned $X(t) = y$. Clearly*

$$f_{s|t}(x|y) = \frac{f_s(x) f_{t-s}(y-x)}{f_t(y)} = C \cdot \exp\left(-\frac{(x - ys/t)^2}{2s(t-s)/t}\right),$$

where C is some universal constant irrelevant to x, y, s, t. As a result, the conditional distribution is the Gaussian $N(\frac{s}{t}y, \frac{s}{t}(t-s))$.

Let $\{W(t)\}_{t\ge 0}$ be a a standard Brownian motion. If $\{X(t)\}_{t\ge 0}$ satisfies $X(t) = \mu \cdot t + \sigma W(t)$, we call $\{X(t)\}_{t \ge 0}$ a (μ, σ^2) Brownian motion. Clearly, $X(t) \sim N(\mu t, \sigma^2 t)$.

2 *The Hitting Time of a Brownian Motion*

We consider the first arrivial time of position b in a Brownian motion. This is called the *hitting time* of b. Let us first consider the standard Brownian motion $\{W(t)\}$. Define $\tau_b \triangleq \inf\{t \ge 0 \mid W(t) > b\}$. For any $t > 0$,

$$\begin{aligned}\mathbf{Pr}[\tau_b < t] &= \mathbf{Pr}[\tau_b < t \land W(t) > b] + \mathbf{Pr}[\tau_b < t \land W(t) < b] \\ &= \mathbf{Pr}[W(t) > b] + \mathbf{Pr}[W(t) < b \mid \tau_b < t] \cdot \mathbf{Pr}[\tau_b < t].\end{aligned}$$

This is called the *principle of reflection* of a standard Brownian motion.

Note that $W(t) \sim \mathcal{N}(0, t)$. Let Φ be the cumulative distribution function of standard Gaussian distribution, that is, $\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}}\,dt$. Then

$$\mathbf{Pr}[W(t) > b] = \mathbf{Pr}\left[\frac{W(t)}{\sqrt{t}} > \frac{b}{\sqrt{t}}\right] = 1 - \Phi\left(\frac{b}{\sqrt{t}}\right).$$

Assuming we have known the value of τ_b and $\tau_b < t$, we can regard $\{W(t)\}_{t \geq \tau_b}$ as a Brownian motion starting from b. Thus, as Figure 1 shows, $\mathbf{Pr}[W(t) < b \mid \tau_b < t] = \frac{1}{2}$.

By direct calculation, we have $\mathbf{Pr}[\tau_b < t] = 2\left(1 - \Phi\left(\frac{b}{\sqrt{t}}\right)\right)$.

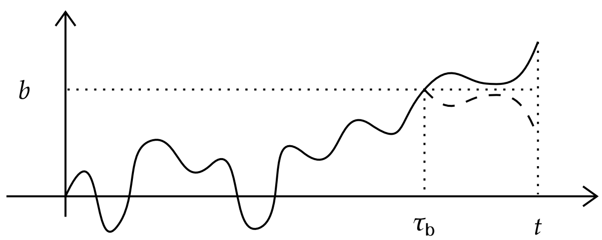

Figure 1: The hitting time and the reflection principle

It is more challenging to find the hitting time of a (μ, σ^2) Brownian motion. The main difficulty is that the principle of reflection no long holds when a nonzero drift μ is present.

We can overcome the difficulty by leveraging the following useful lemma.

Lemma 2. *Let $Y_1, \ldots, Y_n$ be i.i.d. $N(\theta, \nu^2)$ random variables. Then the distribution of $(Y_1, \ldots, Y_n)$ conditioned on $\sum_{i=1}^n Y_i = y$ is irrelevant to θ.*

Proof. Let $X = \sum_{i=1}^n Y_i$. We use $f_{Y_1,\ldots,Y_n|X}$ to denote the density of $Y_1, \ldots, Y_n$ conditioned on X. Then

$$\begin{aligned} f_{Y_1,\ldots,Y_n|X}(y_1, \ldots, y_n, x) &= \frac{f_{Y_1,\ldots,Y_n,X}(y_1, \ldots, y_n, x)}{f_X(x)} \\ &= \frac{f_{Y_1,\ldots,Y_n}(y_1, \ldots, y_{n-1}, x - \sum_{i=1}^{n-1} y_i)}{f_X(x)} \\ &\sim \frac{\exp\left(-\frac{(x - \sum_{i=1}^{n-1} y_i - \theta)^2}{2\nu^2}\right) \prod_{i=1}^{n-1} \exp\left(-\frac{(y_i - \theta)^2}{2\nu^2}\right)}{\exp\left(-\frac{(x - n\theta)^2}{2n\nu^2}\right)}. \end{aligned}$$

A direct calculation shows that all terms on θ cancel and therefore the lemma is proved. □

Here $A \sim B$ means $A = c \cdot B$ for some universal constant c.

The following corollary is immediate since all relevant random variables can be expressed as the sum of independent Gaussians.

Corollary 3. *Let $\{X(t)\}_{t \geq 0}$ be a (μ, σ^2) Brownian motion. Conditioned on $X(t) = x$, for any $t_1 \leq t_2 \ldots t_n \leq t$, the joint distribution of $(X(t_1), X(t_2), \ldots, X(t_n))$ is the same for all μ.*

Armed with this, we can calculate the hitting time of a (μ, σ^2) Brownian motion.

Lemma 4. *Let $X(t)$ be a (μ, σ^2) Brownian motion. For any $y > x$,* $\mathbf{Pr}\left[\tau_y \le t \,\middle|\, X(t) = x\right] = e^{-\frac{2y(y-x)}{t\sigma^2}}$.

Proof. Applying Corollary 3, we know that $\mathbf{Pr}\left[\tau_y \le t \,\middle|\, X(t) = x\right] = \mathbf{Pr}\left[\tau'_y \le t \,\middle|\, X'(t) = x\right]$ where $X'(t)$ is a $N(0, \sigma^2)$ Brownian motion and τ'_y is the hitting time $X'(t)$.

It is not completely rigorous here since Corollary 3 only applies to the joint distribution of *finite* many random variables. Nevertheless, it is conceivable that the same holds for the whole process $X(t)$.

Consider an *infinitesimal change* $\mathrm{d}x$. It holds that

$$\mathbf{Pr}\left[\tau'_y \le t \,\middle|\, X'(t) \in [x, x + \mathrm{d}x]\right] = \frac{\mathbf{Pr}\left[\tau'_y \le t \wedge X'(t) \in [x, x + \mathrm{d}x]\right]}{\mathbf{Pr}\left[X'(t) \in [x + \mathrm{d}x]\right]}.$$

Since $\mathbf{Pr}\left[X'(t) \in [x + \mathrm{d}x]\right] = f_{X'(t)}(x)\mathrm{d}x$, we only need to calculate the numerator. Note that

$$\mathbf{Pr}\left[\tau'_y \le t \wedge X'(t) \in [x, x + \mathrm{d}x]\right] = \mathbf{Pr}\left[\tau'_y \le t\right] \cdot \mathbf{Pr}\left[X'(t) \in [x, x + \mathrm{d}x] \,\middle|\, \tau'_y \le t\right].$$

Applying the principle of reflection, the above is equal to

$$\begin{aligned}\mathbf{Pr}\left[\tau'_y \le t\right] \cdot \mathbf{Pr}\left[X'(t) \in [2y - x - \mathrm{d}x, 2y - x] \,\middle|\, \tau' y \le t\right] &= \mathbf{Pr}\left[X'(t) \in [2y - x - \mathrm{d}x, 2y - x] \wedge \tau'_y \le t\right] \\ &= \mathbf{Pr}\left[X'(t) \in [2y - x - \mathrm{d}x, 2y - x]\right] \\ &= f_{X'(t)}(2y - x)\mathrm{d}x\end{aligned}$$

The second equality is due to that $\mathrm{d}x$ is infinitesimal and therefore $x + \mathrm{d}x < y$. As a result, we have

$$\mathbf{Pr}\left[\tau_y \le t \,\middle|\, X'(t) = x\right] = \frac{f_{X'(t)}(2y - x)}{f_{X'(t)}(x)} = e^{-\frac{2y(y-x)}{t\sigma^2}}.$$

□

We are now ready to compute the hitting time τ_y. When $y \le x$, clearly $\mathbf{Pr}\left[\tau_y \le t \,\middle|\, X(t) = y\right] = 1$. Therefore,

$$\begin{aligned}\mathbf{Pr}\left[\tau_y \le t\right] &= \int_{-\infty}^{\infty} \mathbf{Pr}\left[\tau_y \le t \,\middle|\, X(t) = x\right] \cdot f_{X(t)}(x)\,\mathrm{d}x \\ &= \int_{-\infty}^{y} \mathbf{Pr}\left[\tau_y \le t \,\middle|\, X(t) = x\right] \cdot f_{X(t)}(x)\,\mathrm{d}x + \mathbf{Pr}\left[X(t) \ge y\right] \\ &= \int_{-\infty}^{y} e^{-\frac{2y(y-x)}{t\sigma^2}} \cdot \frac{1}{\sqrt{2\pi t\sigma^2}} e^{-\frac{(x-\mu t)^2}{2t\sigma^2}}\,\mathrm{d}x + \left(1 - \Phi\left(\frac{y - \mu t}{\sigma\sqrt{t}}\right)\right) \\ &= e^{\frac{2y\mu}{\sigma^2}}\left(1 - \Phi\left(\frac{\mu t + y}{\sigma\sqrt{t}}\right)\right) + \left(1 - \Phi\left(\frac{y - \mu t}{\sigma\sqrt{t}}\right)\right).\end{aligned}$$