[AI2613 Lecture 14] Itô Integral, Itô Formula

June 9, 2023

1 Itô Integral

Recall that in the last lecture, we formalized a diffusion $\{X(t)\}$ as

$$\mathrm{d}X(t) = \mu\left(t, X(t)\right)\,\mathrm{d}t + \sigma\left(t, X(t)\right)\,\mathrm{d}W(t),$$

where $\{W(t)\}$ is a standard Brownian motion[1]. With this formalization, the motion of $\{X(t)\}$ in a tiny time interval $[t, t+h]$ can be view as a moving under the differential equation $\frac{\mathrm{d}X(t)}{\mathrm{d}t} = \mu\left(t, X(t)\right)\,\mathrm{d}t$ with a random perturbation $\sigma\left(t, X(t)\right)\,\mathrm{d}W(t)$. In this lecture, we define the mathematical meaning of the above stochastic differential equation.

[1] Here we generalize the specification in the last lecture and allow μ and σ^2 to be functions of two variables, time t and position $X(t)$.

Given an ordinary differential equation $\mathrm{d}f(t) = f(t)\,\mathrm{d}t$, we have that,

$$\forall T,\ \int_0^T \mathrm{d}f(t) = \int_0^T f(t)\,\mathrm{d}t,$$

which is equivalent to

$$\forall T,\ f(t) = f(0) + \int_0^T f(t)\,\mathrm{d}t.$$

If we apply the same process to the stochastic differential equation

$$\mathrm{d}X(t) = \mu\left(t, X(t)\right)\,\mathrm{d}t + \sigma\left(t, X(t)\right)\,\mathrm{d}W(t),$$

we have that

$$\forall T,\ X(T) = X(0) + \int_0^T \mu\left(t, X(t)\right)\,\mathrm{d}t + \int_0^T \sigma\left(t, X(t)\right)\,\mathrm{d}W(t). \tag{1}$$

If we regard $\{X(t)\}$ as a function defined on the sample space Ω, for a fixed $\omega \in \Omega$, $\{X_\omega(t)\}$ and $\{W_\omega(t)\}$ are fixed paths. Then Equation (1) means $\forall T, \forall \omega \in \Omega$,

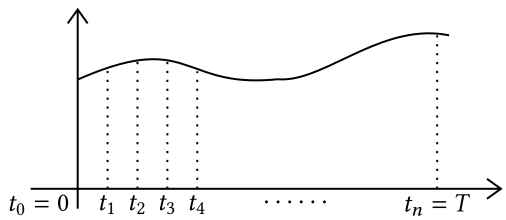
$$X_\omega(T) = X_\omega(0) + \int_0^T \mu\left(t, X_\omega(t)\right)\,\mathrm{d}t + \int_0^T \sigma\left(t, X_\omega(t)\right)\,\mathrm{d}W_\omega(t).$$

Note that $\int_0^T \mu\left(t, X_\omega(t)\right)\,\mathrm{d}t$ is the ordinary Riemann integral of $\mu\left(t, X_\omega(t)\right)$. The main goal today is to rigorously define the meaning of $\int_0^T \sigma\left(t, X_\omega(t)\right)\,\mathrm{d}W_\omega(t)$. Our first try is the Riemann-Stieltjes integral.

1.1 Riemann-Stieltjes Integral

Recall that when we define the Riemann integral of function g on $[0, T]$, we divide the interval into n segments $[t_0, t_1], [t_1, t_2], \dots, [t_{n-1}, t_n]$ where each $t_i - t_{i-1} \to 0$ when $n \to \infty$. Then the Riemann integral is defined by

$$\int_0^T g(t)\,\mathrm{d}t = \lim_{n\to\infty} \sum_{i=1}^n g(t_i^*)\left(t_i - t_{i-1}\right),$$



where t_i^* can be an arbitrary point in $[t_{i-1}, t_i]$.

Let $F\colon [0,T] \to \mathbb{R}$ be a nice enough function. Assume notations defined above. Then the Riemann-Stieltjes integral of g with respect to F is defined by

$$\int_0^T g(t)\,\mathrm{d}F(t) \triangleq \lim_{n\to\infty} \sum_{i=1}^n g(t_i^*)\,(F(t_i) - F(t_{i-1}))\,.$$

When F is the standard Brownian motion $\{W(t)\}$, the Riemann-Stieltjes integral indicates that

$$\int_0^T g(t)\,\mathrm{d}W(t) = \lim_{n\to\infty} \sum_{i=1}^n g(t_i^*)\,(W(t_i) - W(t_{i-1}))\,.$$

However, since the Brownian motion is not a function of bounded variation, $\sum_{i=1}^n g(t_i^*)\,(W(t_i) - W(t_{i-1}))$ may not be convergent[2]. Thus, we cannot find a random variable Y such that for almost every $\omega \in \Omega$, $Y(\omega) = \lim_{n\to\infty} \sum_{i=1}^n g(t_i^*)\,(W_\omega(t_i) - W_\omega(t_{i-1}))$. This indicates that $\int_0^T g(t)\,\mathrm{d}W(t)$ is not well-defined with the Riemann-Stieltjes integral.

The Riemann-Stieltjes integral is commonly used in probability theory. Let X be a random variable on sample space $[0,1]$. Assume that the CDF of X is F and the PDF of X is f. Then the expectation of X is $\mathbf{E}[X] = \int_0^1 X\,dF(t)$. By the definition of the Riemann-Stieltjes integral, we have

$$\int_0^1 X\,dF = \lim_{n\to\infty} X(t_i^*)\,(F(t_i) - F(t_{i-1}))$$
$$= \lim_{n\to\infty} X(t_i^*) f(t_{i-1})(t_i - t_{i-1}) + o(1),$$

which yields that $\mathbf{E}[X] = \int_0^1 X(t)\,dF(t) = \int_0^T X(t)f(t)\,dt$.

[2] It might be convergent for some specific functions g. However, this does not hold in general.

1.2 Itô Integral

Consider the example of $\int_0^T W(t)\,\mathrm{d}W(t)$. Let $\Delta_i W \triangleq W(t_i) - W(t_{i-1})$ and $S_n \triangleq \sum_{i=1}^n W(t_{i-1})\Delta_i W$. By direct calculation, we have that

$$S_n = \frac{1}{2}W_T^2 - \frac{1}{2}\sum_{i=1}^n (\Delta_i W)^2\,.$$

Let $Q_n \triangleq \sum_{i=1}^n (\Delta_i W)^2$. Note that $\Delta_i W \sim \mathcal{N}(0, \Delta_i)$ where $\Delta_i \triangleq t_i - t_{i-1}$. Then

$$\mathbf{E}[Q_n] = \sum_{i=1}^n \mathbf{E}\left[(\Delta_i W)^2\right] = \sum_{i=1}^n \Delta_i = T,$$

and

$$\begin{aligned}\mathbf{Var}[Q_n] &= \sum_{i=1}^n \mathbf{Var}\left[(\Delta_i W)^2\right] = \sum_{i=1}^n \mathbf{E}\left[(\Delta_i W)^4\right] - \sum_{i=1}^n \mathbf{E}\left[(\Delta_i W)^2\right]^2 \\ &= 2\sum_{i=1}^n \Delta_i^2 \le 2\left(\max_{i\in[n]} \Delta_i\right)\cdot \sum_{i=1}^n \Delta_i \overset{n\to\infty}{\longrightarrow} 0.\end{aligned}$$

Note that $\mathbf{Var}[Q_n] = \mathbf{E}\left[(Q_n - \mathbf{E}[Q_n])^2\right] = \mathbf{E}\left[(Q_n - T)^2\right]$, and therefore $\mathbf{E}\left[(Q_n - T)^2\right] \to 0$ as $n \to \infty$. This means that Q_n converges to T is the following mean square sense.

Definition 1 (Mean Square Convergence). *Let $Z_1, Z_2, \ldots$ and Z be random variables that $\mathbf{E}\left[Z^2\right] < \infty$ and $\mathbf{E}\left[Z_n^2\right] < \infty$ for $n \geq 1$. We say $Z_n \to Z$ in mean square, or Z is the mean square limit of $\{Z_n\}$, if $\lim_{n\to\infty} \mathbf{E}\left[(Z_n - Z)^2\right] = 0$.*

Itô integral is defined in a similar way.

Definition 2. *Assume that $\{X(t)\}$ is a "nice enough" stochastic process [3]. Then we define the integral $\int_0^T X(t)\,\mathrm{d}W(t)$ as the mean square limit of*

$$\sum_{i=1}^{n} X(t_{i-1})\left(W(t_i) - W(t_{i-1})\right).$$

This is called the Itô integral of $\{X(t)\}$ with respect to $\{W(t)\}$.

[3] This means that the process has some nice properties such as $X(t)$ is independent with $\{W(u)\}_{u>t}$ for all t. Some other technical requirements can be found in any standard textbook on stochastic differential equations (e.g., [Kle12])

With Definition 2, we can verify that

$$\int_0^T W(t)\,\mathrm{d}W(t) = \frac{1}{2}W_T^2 - \frac{1}{2}T.$$

More generally, we may define $\int_0^T X(t)\,\mathrm{d}W(t)$ as the mean square limit of

$$\sum_{i=1}^{n} X(t_i^*)\left(W(t_i) - W(t_{i-1})\right)$$

where $t_i^* = \alpha t_{i-1} + (1-\alpha)t_i$ with $\alpha \in [0,1]$. The Itô integral defined in Definition 2 corresponds to the case that $\alpha = 1$. By choosing $\alpha = \frac{1}{2}$, we have the definition of Stratonovich integral and it holds that $\int_0^T W(t)\,\mathrm{d}W(t) = \frac{1}{2}W_T^2$ with Stratonovich integral.

Since we can define integrals for all $\alpha \in [0,1]$, it is natural to ask which value of α produces the "correct" one? Actually, the best choice of α depends on how you want to model the stochastic process. For example, when we view a diffusion $\{X_t\}$ as the limit of a certain discrete process, the motion during $[t, t+h]$ for a tiny h is specified by $\mu(t, X_t)$ and $\sigma^2(t, X_t)$. So it is reasonable to specify a diffusion with Itô integral. However, for many stochastic processes from physics which are continuous in nature, it turns out that Stratonovich integral fits better. See discussions in [MM22].

2 Itô Formula

Recall that in the example in Section 1.2, we have

$$Q_n = \sum_{i=1}^{n}\left(W(t_i) - W(t_{i-1})\right)^2, \quad \mathbf{E}\left[Q_n\right] = T \quad \text{and} \quad \mathbf{Var}\left[Q_n\right] \stackrel{n\to\infty}{\longrightarrow} 0.$$

Note that when $n \to \infty$, $\Delta_i W = \mathrm{d}W(t_{i-1})$. Then

$$\forall T, \int_0^T \left(\mathrm{d}W(t_{i-1})\right)^2 = T = \int_0^T \mathrm{d}t.$$

This suggests that $(\mathrm{d}W(t))^2 \approx \mathrm{d}t$ holds. By definition of $\{W(t)\}$, we know that $\mathrm{d}W(t) = W(t+\mathrm{d}t) - W(t) \sim \mathcal{N}(0, \mathrm{d}t)$. Let X and Y be two random variables that $X \sim \mathcal{N}(0, \mathrm{d}t)$ and $Y = X^2$. Then the formula $(\mathrm{d}W(t))^2 \approx \mathrm{d}t$ tells us that Y is well concentrated on $\mathrm{d}t$.

With this observation, we then (heuristically) deduce the chain rule under the definition of Itô integral.

2.1 Classical Chain Rule of Differentiation

In the classical context, for any two differentiable functions g and f, the chain rule of differentiation is

$$\frac{\mathrm{d}f(g(t))}{\mathrm{d}t} = f'(g(t)) \cdot g'(t).$$

This can be verified using the Taylor expansion:

$$\begin{aligned} \mathrm{d}f(g(t)) &= f(g(t+\mathrm{d}t)) - f(g(t)) \\ &= f(g(t) + \mathrm{d}g(t)) - f(g(t)) \\ &= f'(g(t))\,\mathrm{d}g(t) + \frac{1}{2}f''(g(t))\,(\mathrm{d}g(t))^2 + \frac{1}{6}f'''(g(t))\,(\mathrm{d}g(t))^3 + o((\mathrm{d}g(t))^3). \end{aligned}$$

Then it follows that

$$\frac{\mathrm{d}f(g(t))}{\mathrm{d}t} = f'(g(t)) \cdot g'(t) + o(\mathrm{d}t) = f'(g(t)) \cdot g'(t),$$

where $\mathrm{d}t$ tends to 0.

2.2 The Chain Rule with Itô Integral

Similarly, we have

$$\begin{aligned} \mathrm{d}f(W(t)) &= f(W(t) + \mathrm{d}W(t)) - f(W(t)) \\ &= f'(W(t))\,\mathrm{d}W(t) + \frac{1}{2}f''(W(t))\,(\mathrm{d}W(t))^2 + \frac{1}{6}f'''(W(t))\,(\mathrm{d}W(t))^3 + o((\mathrm{d}W(t))^3). \end{aligned}$$

This yields the Itô formula

$$\mathrm{d}f(W(t)) = f'(W(t))\,\mathrm{d}W(t) + \frac{1}{2}f''(W(t))\,\mathrm{d}t,$$

as $\mathrm{d}t \to 0$.

Consider a diffusion $\{X(t)\}$ that

$$\mathrm{d}X(t) = \mu(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}W(t).$$

By the Itô formula, we have

$$\begin{aligned} \mathrm{d}f(X(t)) &= f(X(t) + \mathrm{d}X(t)) - f(X(t)) \\ &= f'(X(t))\,\mathrm{d}X(t) + \frac{1}{2}f''(X(t))\,(\mathrm{d}X(t))^2 \\ &= f'(X(t))\,\mu(t, X(t))\,\mathrm{d}t + f'(X(t))\,\sigma(t, X(t))\,\mathrm{d}W(t) + \frac{1}{2}f''(X(t))\,(\sigma(t, X(t)))^2\,\mathrm{d}t \end{aligned}$$

Then we see some examples of Itô formula.

Example 1 (Geometric Brownian Motion). *Recall the geometric Brownian motion $Y(t) = f(X(t))$ where f is the exponential function and $\{X(t)\}$ is a diffusion specified by $\mu(t, X(t)) \equiv 0$ and $\sigma(t, X(t)) \equiv 1$. Then it follows from the Itô formula that*

$$\mathrm{d}Y(t) = e^{X(t)}\left(\mathrm{d}W(t) + \frac{1}{2}\,\mathrm{d}t\right) = Y(t)\,\mathrm{d}W(t) + \frac{Y(t)}{2}\,\mathrm{d}t.$$

Example 2 (Ornstein-Uhlenbeck Process). *Let $\{X(t)\}$ be a Ornstein-Uhlenbeck process that $\mathrm{d}X(t) = -X(t)\,\mathrm{d}t + 2\,\mathrm{d}W(t)$. Let $f(t,x) = e^t \cdot x$. We adopt the following notations:*

$$f_t(t_0, x_0) = \frac{\partial f(t,x)}{\partial x}\bigg|_{t=t_0, x=x_0},$$
$$f_{tx}(t_0, x_0) = \frac{\partial}{\partial t}\frac{\partial}{\partial x} f(t,x)\bigg|_{t=t_0, x=x_0}.$$

and similarly define $f_x(t_0,x_0)$, $f_{xx}(t_0,x_0)$, $f_{tt}(t_0,x_0)$, and $f_{xt}(t_0,x_0)$. Then $f(t,X(t)) = e^t \cdot X(t)$ and using Taylor expansion, we have

$$\begin{aligned}
\mathrm{d}f(t,X(t)) &= f(t+\mathrm{d}t, X(t+\mathrm{d}t)) - f(t,X(t)) \\
&= f_t(t,X(t))\,\mathrm{d}t + f_x(t,X(t))\,\mathrm{d}X(t) \\
&\quad + \frac{1}{2}\left(f_{tt}(t,X(t))(\mathrm{d}t)^2 + 2f_{tx}(t,X(t))\,\mathrm{d}t\,\mathrm{d}X(t) + f_{xx}(t,X(t))\,(\mathrm{d}X(t))^2\right) \\
&= f_t(t,X(t))\,\mathrm{d}t + f_x(t,X(t))\,\mathrm{d}X(t) + \frac{1}{2} f_{xx}(t,X(t))\,(\mathrm{d}X(t))^2 + o(\mathrm{d}t)
\end{aligned}$$

Note that $f_{xx}(t,X(t)) = 0$. Thus

$$\begin{aligned}
\mathrm{d}f(t,X(t)) &= f_t(t,X(t))\,\mathrm{d}t + f_x(t,X(t))\,\mathrm{d}X(t) \\
&= e^t X(t)\,\mathrm{d}t + e^t\,\mathrm{d}X(t) \\
&= 2e^t\,\mathrm{d}W(t).
\end{aligned}$$

This yields that $\mathrm{d}(e^t X(t)) = 2e^t\,\mathrm{d}W(t)$ and sequentially

$$X(T) = e^{-T}\left(\int_0^T 2e^t\,\mathrm{d}W(t) + X(0)\right).$$

References

[Kle12] Fima C Klebaner. *Introduction to stochastic calculus with applications.* World Scientific Publishing Company, 2012. 3

[MM22] Riccardo Mannella and Peter VE McClintock. Itô versus stratonovich: 30 years later. *The Random and Fluctuating World: Celebrating Two Decades of Fluctuation and Noise Letters*, pages 9–18, 2022. 3