[AI2613 Lecture 8] Optional Stopping Theorem

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1 Stopping Time

Suppose $Z_0, Z_1, \ldots, Z_n, \ldots$ is a martingale with respect a certain filtration $\{\mathcal{F}_t\}$. We know that for any t, $\mathbf{E}[Z_t] = \mathbf{E}[Z_0]$. However, does $\mathbf{E}[Z_\tau] = \mathbf{E}[Z_0]$ still hold if τ is a random variable?

Consider the following gambling strategy in a fair game. At the first round, the gambler bet \$1. Then he simply double his stake until he wins

The strategy was called martingale!

Let τ be the first time he wins. Then expected money he win at time τ is 1, which is not equal to 0, his initial money. In order to understand the phenomenon, let us first formally introduce *stopping time*.

- If $\tau = 1$, he wins 1 dollar.
- If $\tau = 2$, he wins $-1 + 2 = 1$ dollar.
- If $\tau = 3$, he wins $-1 - 2 + 4 = 1$ dollar.
- $\ldots$

Definition 1 (Stopping Time) *Let $\tau \in \mathbb{N} \cup \{\infty\}$ be a random variable. We say τ is a* stopping time *if for all $t \geq 0$, the event "$\tau \leq t$" is $\mathcal{F}_t$-measurable.*

For example, the first time that a gambler wins five games in a row is a stopping time, since for a given t, this can be determined by looking at the outcomes of all the previous games, and therefore the time is $\mathcal{F}_t$-measurable. However, the *last* time the gambler wins five games in a row is *not* a stopping time, since determining whether the time is t cannot be done without knowing $X_{t+1}, X_{t+2}, \ldots$

1.1 Optional Stopping Theorem(OST)

The optional stopping theorem provides sufficient condition for $\mathbf{E}[Z_\tau] = \mathbf{E}[Z_0]$ to hold.

Theorem 2 (Optional Stopping Theorem) *Let $\{X_t\}_{t\geq 0}$ be a martingale and τ be a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Then $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ if* at least one of *the following conditions holds:*

1. *τ is bounded almost surely, that is, $\exists n \in \mathbb{N}$ such that $\mathbf{Pr}[\tau \leq n] = 1$;*
2. *$\mathbf{Pr}[\tau < \infty] = 1$, and there is a finite M such that $|X_t| \leq M$ for all $t < \tau$;*
3. *$\mathbf{E}[\tau] < \infty$, and there is a constant c such that $\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \leq c$ for all $t < \tau$.*

We will prove the theorem next time. Let us look back at the boy-or-girl example mentioned in the first class.

Example 1 (Boy or Girl) *Suppose there is a country in which people only want boys. What is the sex ratio of the country in the following three scenarios?*

- *Each family continues to have children until they have a boy.*

- *Each family continues to have children until there are more boys.*
- *Each family continues to have children until there are more boys or there are* 10 *children.*

We can model the problem as a random walk. Suppose there is a man walking randomly on a one-dimensional axis. Let $\{X_t\}_{t\ge 0}$ *be the positions of the man at each time where* X_t *stands for* the number of boys minus the number of girls *in the first* t *children of a family. Starting at* $X_0 = 0$*, at time* 0*, the man takes a step* $c_t \in_{\mathrm{R}} \{-1, 1\}$ *and reach* X_{t+1}*, i.e.,* $X_{t+1} = X_t + c_t$*. It is easy to verify that* $\{X_t\}_{t\ge 0}$ *is a martingale. The three scenarios mentioned correspond to the following three different definitions of a stopping time* τ*. The identity* $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ *means that the sex ratio is balanced. We will check respectively whether it is the case using OST.*

- *Let* τ *be the first time* t *such that* $c_t = 1$*. Then* $\mathbf{E}[\tau] < \infty$ *since by definition* $\tau \sim \mathrm{Geom}\left(\frac{1}{2}\right)$*, and* $|X_{t+1} - X_t| \le 1$ *for all* $t < \tau$*. Therefore from the 3rd condition of OST we have* $\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = 0$*. In other words, if the man stops at the first time of* $c_t = 1$*, then the expected final position is* 0*.*
- *Let* τ *be the first time* t *such that* $X_t = 1$*, then of course* $\mathbf{E}[X_\tau] = 1 \neq \mathbf{E}[X_0]$*. This process is called "*1*-d random walk with one absorbing barrier" and it is well-known that* $\mathbf{E}[\tau] = \infty$*. No condition in OST is satisfied.*
- *Let* τ *be the minimum between* 10 *and the first time* t *such that* $X_t = 1$*. In this case,* τ *is at most* 10*, which satisfies the first condition of OST. Therefore we have* $\mathbf{E}[X_\tau] = \mathbf{E}[X_0] = 0$*.*

The property $\mathbf{E}[\tau] = \infty$ of the random work is called "null recurrent". You can find more on this from my lecture on stochastic processes.

2 Applications of OST

2.1 Doobs martingale inequality

With OST, we can obtain concentration property of the maximum element in a seuqence of random variables.

Claim 3 *Let* $\{X_t\}_{t\ge 0}$ *be a martingale with respect to itself where* $X_t \ge 0$ *for every* t*. Prove that for every* $n \in \mathbb{N}$*,*

$$\mathbf{Pr}\left[\max_{0\le t\le n} X_t \ge \alpha\right] \le \frac{\mathbf{E}[X_0]}{\alpha}.$$

Proof. We define a stopping time τ when the first element that is greater that α occurs, and otherwise set $\tau = n$. Formally, define

$$\tau \triangleq \min\left(n, \min_{t\le n}\{t \mid X_t \ge \alpha\}\right).$$

By definition of τ, we have

$$\mathbf{Pr}\left[\max_{0\le t\le n} X_t \ge \alpha\right] = \mathbf{Pr}[X_\tau \ge \alpha].$$

Since τ is bounded, we apply Optional Stopping Theorem to obtain that $E[X_\tau] = E[X_0]$. Therefore, by Markov's Inequality,

$$\mathbf{Pr}\left[\max_{0\le t\le n} X_t \ge \alpha\right] = \mathbf{Pr}\left[X_\tau \ge \alpha\right] \le \frac{\mathbf{E}\left[X_\tau\right]}{\alpha} = \frac{\mathbf{E}\left[X_0\right]}{\alpha}$$

□

2.2 One-dimensional Random Walk with Two Absorbing Barriers

We've discussed one-dimensional random walk with one absorbing barrier before

We consider another problem in one-dimensional random walk. Let $a, b > 0$ be two integers. A man starts the random walk from 0 and stops when he arrives at $-a$ or b. Let τ be the time when the man first reaches $-a$ or b, i.e., the first time t that $X_t = -a$ or $X_t = b$. The model is called "one-dimensional random walk with two absorbing barriers". We want to compute the expected value of $\mathbf{E}\left[\tau\right]$, that is, the average stopping time of the walk.

We want to construct a martingale $\{Y_t\}_{t\ge 0}$ such that OST can be applied to $\{Y_t\}_{t\ge 0}$ and τ and thereby we can derive an equality related to $\mathbf{E}\left[\tau\right]$. Before calculating $\mathbf{E}\left[\tau\right]$, we first determine $\mathbf{Pr}\left[X_\tau = -a\right]$, the probability that the man stops at position $-a$. Let $P_a \triangleq \mathbf{Pr}\left[X_\tau = -a\right]$. We want to apply OST to show $\mathbf{E}\left[X_\tau\right] = \mathbf{E}\left[X_0\right]$. Therefore, we verify that some of conditions in OST is satisfied.

In a time period of length $T = a + b$, if the man walks towards the same direction, he must have stopped, either at $-a$ or b, which happens with probability $2^{-(a+b)}$. Therefore, if we divide the time into consecutive periods in this manner, in expected finite time, we can meet some period when the event happened. Hence, $\mathbf{E}\left[\tau\right] < \infty$. Moreover, we clearly have $\mathbf{E}\left[|X_{t+1} - X_t| \mid \mathcal{F}_t\right] < 1$ for every $0 \le t < \tau$, so the third condition of OST holds, which implies that $\mathbf{E}\left[X_\tau\right] = \mathbf{E}\left[X_0\right]$. On the other hand, we have $\mathbf{E}\left[X_\tau\right] = P_a \cdot (-a) + (1 - P_a) \cdot b$. These two equalities give $P_a = \frac{b}{a+b}$.

Then for all $t \ge 0$, we define a new random variable $Y_t \triangleq X_t^2 - t$ which involves the time t. The following fact is easy to verify by definition.

Claim 4 $\{Y_t\}_{t\ge 0}$ *is a martingale.*

Proof. First we have

$$\begin{aligned}
\mathbf{E}\left[Y_{t+1} \mid \mathcal{F}_t\right] &= \mathbf{E}\left[X_{t+1}^2 - (t+1) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbf{E}\left[(X_t + c_t)^2 - (t+1) \,\middle|\, \mathcal{F}_t\right] \\
&= \mathbf{E}\left[X_t^2 \,\middle|\, \mathcal{F}_t\right] + 2\mathbf{E}\left[X_t c_t \mid \mathcal{F}_t\right] + \mathbf{E}\left[c_t^2 \,\middle|\, \mathcal{F}_t\right] - (t+1)\,.
\end{aligned}$$

Since X_t is $\mathcal{F}_t$-measurable, $\mathbf{E}\left[c_t \mid \mathcal{F}_t\right] = 0$ and $\mathbf{E}\left[c_t^2 \,\middle|\, \mathcal{F}_t\right] = 1$, we can further derive that

$$\mathbf{E}\left[Y_{t+1} \mid \mathcal{F}_t\right] = X_t^2 + 0 + 1 - (t+1) = X_t^2 - t = Y_t.$$

Hence $\{Y_t\}_{t\ge 0}$ is a martingale. □

Sometimes one can use OST in a reverse way. Consider the random walk with only one barrier at $-a$. The fact that $\mathbf{E}\left[\tau\right] = \infty$ can be proved in the following way (due to Biaoshuai Tao): If $\mathbf{E}\left[\tau\right] < \infty$, then by (cond 3 of) OST, $\mathbf{E}\left[X_\tau\right] = \mathbf{E}\left[X_0\right] = 0$. On the other hand, we know $X_\tau = -a \neq 0$. Therefore it must be that $\mathbf{E}\left[\tau\right] = \infty$.

Note that $X_t \in [-a, b]$ for all $t \geq 0$. Thus $|Y_{t+1} - Y_t| = \left|X_{t+1}^2 - (t+1) - X_t^2 + t\right| = \left|X_{t+1}^2 - X_t^2 - 1\right|$ is bounded by some constant. We can apply OST again to obtain $\mathbf{E}[Y_\tau] = \mathbf{E}[Y_0] = 0$. On the other hand, we have $\mathbf{E}[Y_\tau] = \mathbf{E}\left[X_\tau^2\right] - \mathbf{E}[\tau]$ by definition, and thus

$$\mathbf{E}[\tau] = \mathbf{E}\left[X_\tau^2\right] = a^2 P_a + b^2(1 - P_a) = a^2 \cdot \frac{b}{a+b} + b^2 \cdot \frac{a}{a+b} = ab.$$

2.3 Pattern Matching

Suppose that there is a $\{H, T\}$-string P of length ℓ (H for "head" and T for "tail"). We flip a coin consecutively until the last ℓ results form exactly the same string as P. How many times do we flip the coin?

Note that if we flip the coin N times and observe the string S consisting of N results. No matter which pattern we choose, by the linearity of expectation, the expected number of occurrence [1] is

$$\mathbf{E}[\#\text{ of occurrence of } P \text{ in } S] = \sum_{i=1}^{n-\ell+1} \mathbf{E}\left[\mathbb{1}[S_{i,i+1,\dots,i+\ell-1} = P]\right] = (n - l + 1)\cdot 2^{-\ell}.$$

[1] That means the expected number of substrings exactly the same as P in the resulting string S.

However, if we would like to compute the first time that pattern P occurs, the pattern itself has an impact on the expected time. Intuitively, lets consider two patterns HT and HH. Assume that the first flipping result is H. Then we consider what happens if the second result fails. Suppose that the desired pattern is HT and H appears. Although we fail, we obtain an H. However, if the desired pattern is HH and the second flipping result is T, then we obtain nothing and the first two flips are a waste. So we should believe that the expected times of the first occurrence of HT is smaller than HH.

We now use the optional stopping theorem to solve this problem. Let $P = p_1 p_2 \dots p_\ell$. For every $n \geq 0$, assume that before $n+1$-th flipping there is a new gambler G_{n+1} coming with 1 unit of money to bet that the following ℓ result (i.e., the $n+1$-th to $n+\ell$-th results) are exactly the same as P. At the $n+k$-th flipping, G_{n+1} will bet that the result is p_k by an all in strategy, that is, if the $n+k$-th result is p_k then G_{n+1} will have twice as much money as before; otherwise they will lose all. Suppose that the patter P = HTHTH and the flipping results are HTHHTHTH. The following table shows the total money of each gambler after flipping.

Let X_t be the result of t-th flipping, $M_i(t)$ denote the money that G_i has after t-th flipping, and $Z_t \triangleq \sum_{i=1}^{t}(M_i(t) - 1)$ be the total income of all gamblers after t-th flipping. It is easy to verify that $\{M_i(t)\}_{t\geq 0}$ is a martingale with respect to $\{X_t\}$ since

$$\mathbf{E}\left[M_i(t+1) \mid \bar{X}_{0,t}\right] = \frac{1}{2} \cdot 2M_i(t) + \frac{1}{2} \cdot 0 = M_i(t).$$

Then by the linearity of expectation we conclude that $\{Z_t\}_{t\geq 0}$ is a martingale with respect to the flipping results $\{X_t\}$ since $\mathbf{E}[M_i(t)] = 1$. Let

Gambler	H	T	H	H	T	H	T	H	Money	
1	H	T	H	T					0	1→2→4→8→0
2		H							0	1→0
3			H	T					0	1→2→0
4				H	T	H	T	H	32	1→2→4→8→16→32
5					H				0	1→0
2						H	T	H	8	1→2→4→8
5							H		0	1→0
5								H	2	1→2

τ be the stopping time defined by the first time that some gambler wins, namely, the first time that P occurs in the flipping results. Applying Condition 2 of OST we obtain that $\mathbf{E}[Z_\tau] = \mathbf{E}[Z_0] = 0$. Sequentially we have $\mathbf{E}\left[\sum_{i=1}^{\tau} M_i(\tau) - \tau\right] = 0$ and $\mathbf{E}[\tau] = \sum_{i=1}^{\tau} \mathbf{E}[M_i(\tau)]$.

Note that $M_i(t) = 0$ for $i \le \tau - \ell$ and $M_i(t) = 2^{\tau-i+1}\chi_{\tau-i+1}$ for $i > \tau - \ell$ where χ_j is defined by

$$\chi_j = \mathbb{1}[p_1p_2 \dots p_j = p_{\ell-j+1} \dots p_{\ell-1}p_\ell].$$

Hence,

$$\mathbf{E}[\tau] = \sum_{i=\tau-\ell+1}^{\tau} \mathbf{E}[M_i(\tau)] = \sum_{i=1}^{\ell} 2^i \chi_i.$$

Recall the example of HH and HT. If P is HH, $\mathbf{E}[\tau] = 2 + 4 = 6$. If P is HT, $\mathbf{E}[\tau] = 4$. This confirms our hypothesis that $\mathbf{E}[\tau]$ for HH is larger than $\mathbf{E}[\tau]$ for HT.

2.4 *Wald's Equation*

In practice, we often need to analyze the (expected) running time of following procedure where both *cond* and *compute()* are random.

```
while cond do
     compute();
end while
```

Assume the i-th call to *compute()* costs X_i time and the algorithm terminates after T iterations. Then the total running time is $N \triangleq \sum_{i=1}^{T} X_i$. Suppose X_is are independently and identically distributed as a random variable X. The Wald's equation gives a formula for $\mathbf{E}[N]$.

Theorem 5 (Wald's Equation) *If we have*

- $X_1, X_2, \dots$ *are non-negative, independent, identically distributed random variables with the same distribution as* X.
- T *is a stopping time for* $X_1, X_2, \dots$.

- $\mathbf{E}[T], \mathbf{E}[X] < \infty$,

then

$$\mathbf{E}\left[\sum_{i=1}^{T} X_i\right] = \mathbf{E}[T] \cdot \mathbf{E}[X].$$

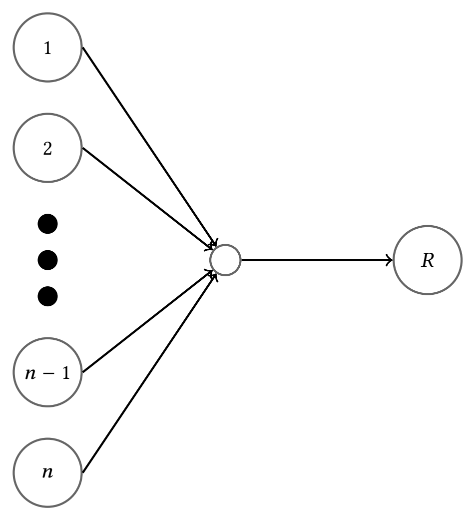
Proof. For $i \geq 1$, let $Z_i := \sum_{j=1}^{i}(X_j - \mathbf{E}[X])$. Clearly the sequence $Z_i, Z_2, \dots$ is a martingale with respect to $X_1, X_2, \dots$ and $\mathbf{E}[Z_1] = 0$. And we have

$$\begin{aligned}\mathbf{E}[|Z_{i+1} - Z_i| \mid \mathcal{F}_i] &= \mathbf{E}[|X_{i+1} - \mathbf{E}[X]| \mid \mathcal{F}_i] \\ &\leq \mathbf{E}[X_{i+1} + \mathbf{E}[X] \mid \mathcal{F}_i] \\ &\leq 2\mathbf{E}[X].\end{aligned}$$

We know that $\mathbf{E}[T], \mathbf{E}[X] < \infty$, and therefore applying OST derives $\mathbf{E}[Z_T] = \mathbf{E}[Z_1] = 0$. Then

$$\begin{aligned}\mathbf{E}[Z_t] &= \mathbf{E}\left[\sum_{j=1}^{T}(X_j - \mathbf{E}[X])\right] \\ &= \mathbf{E}\left[\sum_{i=1}^{T} X_i - T\mathbf{E}[X]\right] \\ &= \mathbf{E}\left[\sum_{i=1}^{T} X_i\right] - \mathbf{E}[T]\,\mathbf{E}[X] = 0.\end{aligned}$$

□



An Application of Wald's Equation: A Routing Problem Let us consider an application of Wald's equation. There are n senders and one receiver. In each round, each sender sends a packet to the receiver with probability $\frac{1}{n}$. Since all senders share the same channel, if there are multiple packets sent at the same time, all of them will fail. The question is, on average, how many rounds are required so that each sender can successfully send at least one packet.

We let X_i be the variable indicating how long the receiver needs to get another packet after he has received $i - 1$ ones (counting packets from repeated sender). And let T be the number of packets received when first time the receiver receives at least one packet from each sender. The quantity we are interested in is

$$N \triangleq \sum_{i=1}^{T} X_i.$$

Clearly $X_1, X_2, \dots$ are independently and identically distributed, and $\mathbf{E}[T]$ is finite. Therefore $\mathbf{E}[N] = \mathbf{E}[T] \cdot \mathbf{E}[X_1]$ by Wald's equation.

Note that by the definition, T is the number of coupons in the coupon collector's problem we met before. So $\mathbf{E}[T] = nH_n = \Theta(n \log n)$.

On the otherhand, $X_1 \sim \mathtt{Geom}(p)$ with

$$p = n \cdot \frac{1}{n}\left(1 - \frac{1}{n}\right)^{n-1} \approx e^{-1}$$

which implies $E[X_1] = e$. Therefore,

$$\mathbf{E}[N] = \mathbf{E}[T] \cdot \mathbf{E}[X_1] \approx enH_n.$$

3 *Proof of Optional Stopping Theorem*

Let us restate the theorem.

Theorem 6 (Optional Stopping Theorem) *Let $\{X_t\}_{t\geq 0}$ be a martingale and τ be a stopping time with respect to $\{\mathcal{F}_t\}_{t\geq 0}$. Then $\mathbf{E}[X_\tau] = \mathbf{E}[X_0]$ if* at least one of *the following conditions holds:*

1. *τ is bounded almost surely, that is, $\exists n \in \mathbb{N}$ such that $\mathbf{Pr}[\tau \leq n] = 1$;*
2. *$\mathbf{Pr}[\tau < \infty] = 1$, and there is a finite M such that $|X_t| \leq M$ for all $t < \tau$;*
3. *$\mathbf{E}[\tau] < \infty$, and there is a constant c such that $\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \leq c$ for all $t < \tau$.*

Proof. It is obvious that for every $n \in \mathbb{N}$, $\mathbf{E}[X_n] = \mathbf{E}[X_0]$. So first we show that for every $n \in \mathbb{N}$, $\mathbf{E}\left[X_{\min\{n,\tau\}}\right] = \mathbf{E}[X_0]$. Define $Z_n \triangleq X_{\min\{n,\tau\}} = X_0 + \sum_{i=0}^{n-1}(X_{i+1} - X_i)\mathbb{1}[\tau > i]$. We verify that $\{Z_n\}_{n\geq 0}$ is a martingale. By definition

$$\begin{aligned}\mathbf{E}[Z_{n+1} \mid \mathcal{F}_n] &= \mathbf{E}[Z_n + (X_{n+1} - X_n)\mathbb{1}[\tau > n] \mid \mathcal{F}_n] \\ &= Z_n + \mathbb{1}[\tau > n]\left(\mathbf{E}[X_{n+1} \mid \mathcal{F}_n] - X_n\right) \\ &= Z_n.\end{aligned}$$

So we have $\mathbf{E}\left[X_{\min\{n,\tau\}}\right] = \mathbf{E}[Z_n] = \mathbf{E}[Z_0] = \mathbf{E}[X_0]$.

Therefore, this motivates us to decompose X_τ into two terms:

$$\forall n \in \mathbb{N},\ X_\tau = X_{\min\{n,\tau\}} + \mathbb{1}[\tau > n] \cdot (X_\tau - X_n).$$

Taking expectation and letting n tend to infinity, we obtain

$$\mathbf{E}[X_\tau] = \mathbf{E}[X_0] + \lim_{n\to\infty} \mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)].$$

Therefore, we only need to verify that each of the three conditions in the statement guarantee $\lim_{n\to\infty} \mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] = 0$.

1. If τ is bounded almost surely, then clearly $\mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] = 0$ for sufficiently large n.
2. In this case,

$$\begin{aligned}\mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] &\leq \mathbf{E}[\mathbb{1}[\tau > n] \cdot (|X_\tau| + |X_n|)] \\ &\leq 2M \cdot \mathbf{E}[\mathbb{1}[\tau > n]] \\ &= 2M \cdot \mathbf{Pr}[\tau > n] \to 0 \text{ as } n \to \infty.\end{aligned}$$

3. In order to apply our bounds on the gap between consecutive X_t, we write

$$\begin{aligned}\mathbb{1}[\tau > n] \cdot (X_\tau - X_n) &= \sum_{t=n}^{\tau-1}(X_{t+1} - X_t) \\ &\le \sum_{t=n}^{\tau-1} |X_{t+1} - X_t| \\ &= \sum_{t=n}^{\infty} |X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t].\end{aligned}$$

Taking expectation on both sides, we have

$$\begin{aligned}\mathbf{E}[\mathbb{1}[\tau > n] \cdot (X_\tau - X_n)] &\le \mathbf{E}\left[\sum_{t=n}^{\infty} |X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t]\right] \\ &= \sum_{t=n}^{\infty} \mathbf{E}[|X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t]] \\ &= \sum_{t=n}^{\infty} \mathbf{E}[\mathbf{E}[|X_{t+1} - X_t| \cdot \mathbb{1}[\tau > t] \mid \mathcal{F}_t]] \\ &= \sum_{t=n}^{\infty} \mathbf{E}[\mathbf{E}[|X_{t+1} - X_t| \mid \mathcal{F}_t] \cdot \mathbb{1}[\tau > t]] \\ &\le \sum_{t=n}^{\infty} c \cdot \mathbf{Pr}[\tau > t],\end{aligned}$$

where the first equality follows from the monotone convergence theorem.

On the other hand, we know $\mathbf{E}[\tau] = \sum_{t=0}^{\infty} \mathbf{Pr}[\tau > t] < \infty$. Therefore, the tail of this sequence, $\sum_{t=n}^{\infty} \mathbf{Pr}[\tau > t] \to 0$ as $n \to \infty$.

□